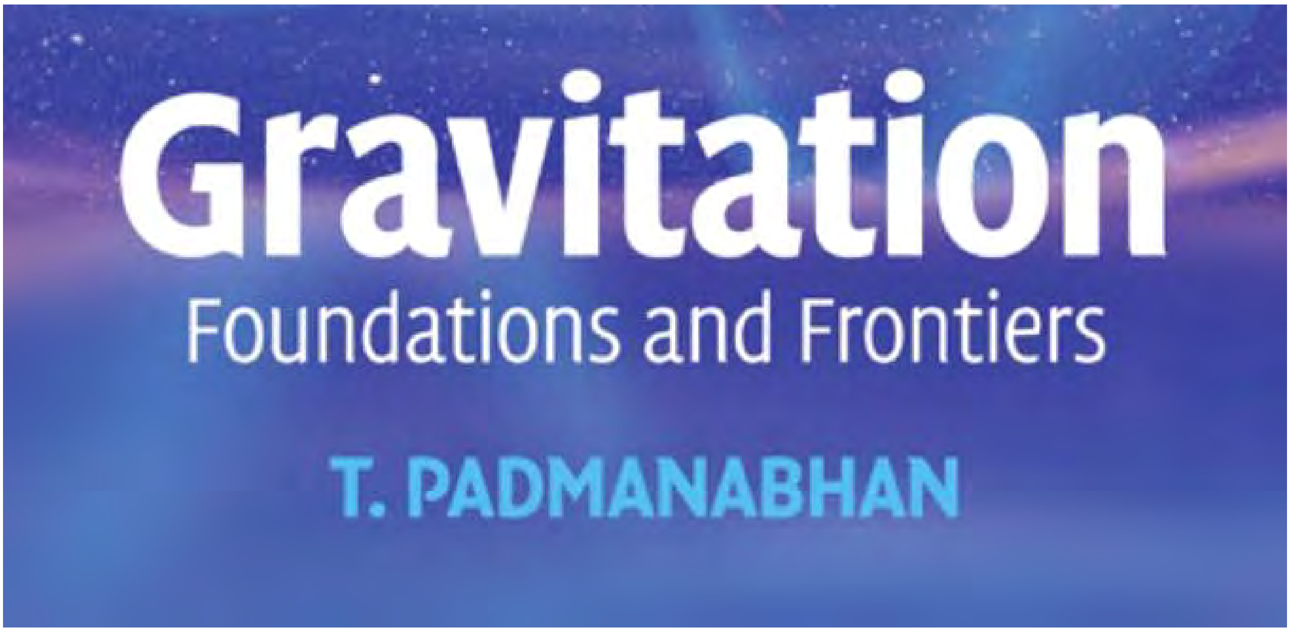

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The equation $\partial_a T^a_b = 0$ in special relativity was interpreted as a conservation law because it could be transformed into a form demonstrating the conservation of a particular quantity. Integrating this equation over a spacetime volume $\mathcal{V}$ and using the Gauss theorem gives:

$$0 = \int_{\mathcal{V}} d^4x \; (\partial_a T^a_b) = \int_{\partial\mathcal{V}} d^3\sigma_a \, T^a_b = \int_{t_2} d^3\boldsymbol{x} \, T^0_b - \int_{t_1} d^3\boldsymbol{x} \, T^0_b. \tag{5.83}$$

The second equality arises from the Gauss theorem and the third arises from the contributions on the surfaces $t =$ constant. As usual, we have ignored the contribution from a surface at spatial infinity assuming that the energy-momentum tensor of the system is confined to a finite region in 3-space. This shows that there exist four quantities

$$Q_b(t) = \int_t d^3\boldsymbol{x} \, T^0_b(t, \boldsymbol{x}) \tag{5.84}$$

which are conserved in the sense that $dQ_b/dt = 0$. It is natural to identify this as the covariant components of the four-momentum of the system with energy-momentum tensor T^a_b.

Consider now placing the above physical system in a gravitational field described by the metric tensor $g_{ab}(x)$. (For example, T^a_b could then describe the energy-momentum tensor of a gas of particles in the Earth's atmosphere.) The gravitational field will influence the system and can, in general, change its energy and momentum. It is unlikely that, in this case, we will have a conserved quantity for the system because it is interacting with an external field. The equation $\partial_k T^k_i = 0$ will now be modified to $\nabla_k T^k_i = 0$. Using Eq. (4.107) we can write this as

$$\frac{1}{\sqrt{-g}}\,\partial_k\left(\sqrt{-g}\,T_i^k\right) = \frac{1}{2}\,(\partial_i g_{kl})\,T^{kl}. \tag{5.85}$$

Integrating both sides over the proper volume $\sqrt{-g}\,d^4x$ we get the result

$$\begin{aligned}\int_{\mathcal{V}} d^4x\partial_k\left(\sqrt{-g}T_i^k\right) &= \int_{t_2} d^3\boldsymbol{x}\sqrt{-g}T_i^0 - \int_{t_1} d^3\boldsymbol{x}\sqrt{-g}T_i^0 \equiv Q_i(t_2) - Q_i(t_1)\\ &= \frac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{-g}\,T^{kl}(\partial_i g_{kl}).\end{aligned} \tag{5.86}$$

The first equality follows from the standard application of the Gauss theorem. This shows that the difference $Q_i(t_2) - Q_i(t_1)$ is nonzero and is related to an integral over a term containing $\partial_i g_{kl}$. As long as $\partial_i g_{kl}$ is nonzero, the right hand side, in

general, will not vanish and $Q_i(t)$ is no longer conserved. As we said before, this is exactly what we will expect in the presence of an external gravitational field.

The analysis also reveals another expected consequence. If g_{kl} is independent of a particular coordinate x^M, say, then Q_M will be conserved. In particular, if g_{ik} is independent of time, the total energy of the system represented by Q_0 will be conserved. This result agrees with what we had already seen, in terms of Killing vectors as the connection between translational invariance of the metric and a conservation law. More formally, given a Killing vector ξ^a in the spacetime, Eq. (4.137) shows that $P^i = T^i_j \xi^j$ is conserved with $\nabla_i P^i = 0$. Using Eq. (4.108) for the covariant derivative of the vector and integrating over $\sqrt{-g}\, d^4x$ we get

$$\begin{aligned} 0 &= \int_{\mathcal{V}} \sqrt{-g}\, d^4x \nabla_i P^i = \int_{\mathcal{V}} d^4x\, \partial_i \left(\sqrt{-g}\, P^i \right) \\ &= \int_{t_2} d^3\boldsymbol{x} \sqrt{-g} P^0 - \int_{t_1} d^3\boldsymbol{x} \sqrt{-g} P^0, \end{aligned} \tag{5.87}$$

which shows that the integral over all space of $\sqrt{-g}\, P^0$ is conserved. A comparison between Eq. (4.107) and Eq. (4.108) clearly shows the structural difference between the expanded forms of equations $\nabla_i T^{ik} = 0$ for a symmetric tensor and $\nabla_i P^i = 0$ for a vector. The former cannot be converted to a pure surface term by using the Gauss theorem (so that the right hand side of Eq. (5.86) is nonzero) while in the latter case we do get a pure surface term. So a vector equation like $\nabla_i P^i = 0$ is a genuine conservation law in curved spacetime but a tensor equation like $\nabla_i T^{ik} = 0$ is not. A Killing vector allows us to convert a tensor equation to a vector equation, thereby leading to a conservation law.

We shall now discuss some explicit examples of the equation $\nabla_i T^{ik} = 0$ starting with the energy-momentum tensor for a single particle. In special relativity, one could have taken the expression to be

$$T^{mn} = m \int \delta_D[x^a - z^a(\tau)]\, u^m u^n \, d\tau, \tag{5.88}$$

where $z^a(\tau)$ is the trajectory of the particle. In this expression, $u^m u^n d\tau$ is generally covariant but not the Dirac delta function. It is, however, easy to see from the relation

$$1 = \int \delta_D(x^a)\, d^4x = \int \frac{\delta_D(x^a)}{\sqrt{-g}} \sqrt{-g}\; d^4x \tag{5.89}$$

that $\delta_D(x)/\sqrt{-g}$ is a scalar. Therefore, the generally covariant definition of the energy-momentum tensor for a single particle is given by

$$\begin{aligned} T^{mn} &= m \int \delta_D[x^a - z^a(\tau)]\, \frac{u^m u^n}{\sqrt{-g}}\, d\tau \\ &= \frac{m}{\sqrt{-g}} \int \delta_D[x^a - z^a(\tau)]\, u^m u^n \, d\tau, \end{aligned} \tag{5.90}$$

where we have pulled out the $\sqrt{-g}$ factor which depends only on x. Using

$$\nabla_m T^{mn} = \frac{1}{\sqrt{-g}} \partial_m \left(\sqrt{-g}\; T^{mn}\right) + \Gamma^n_{sm} T^{ms} \tag{5.91}$$

we can express the relation $\nabla_m T^{mn} = 0$ in the form

$$\int u^m u^n \partial_m \, \delta^4\left(x - z(\tau)\right) d\tau + \Gamma^n_{sm} \int u^m u^s \delta^4\left(x - z(\tau)\right) d\tau = 0. \tag{5.92}$$

Since the Dirac delta function depends only on the difference of the coordinates, we can replace $(\partial/\partial x^m)$ by $-(\partial/\partial z^m)$. Using

$$u^m \frac{\partial}{\partial z^m} \delta^4\left(x - z(\tau)\right) = \frac{d}{d\tau} \delta^4\left(x - z(\tau)\right) \tag{5.93}$$

we get

$$-\int u^n \frac{d}{d\tau} \delta^4\left(x - z(\tau)\right) d\tau + \Gamma^n_{sm} \int u^m u^s \delta^4\left(x - z(\tau)\right) d\tau = 0. \tag{5.94}$$

Doing an integration by parts on the first term, this reduces to

$$\int \left[\frac{du^n}{d\tau} + \Gamma^n_{sm}\, u^m u^s \right] \delta^4 \left(x - z(\tau) \right)\, d\tau = 0. \tag{5.95}$$

The vanishing of this integral requires the expression within the square brackets to vanish along the trajectory of the particle, which is identical to the geodesic equation for the particle. Thus, for a single particle, the condition $\nabla_m T^{mn} = 0$ is equivalent to a geodesic equation. As we have already seen, the geodesic equation is capable of encoding the effect of external gravitational field on a material particle and – in general – will not lead to any conservation law. It follows that the equation $\nabla_m T^{mn} = 0$ describes the way material systems are influenced by external gravity and, of course, is not a conservation law either.