

Finally, we show that a field of differential n-forms can be integrated over an oriented region of an n-dimensional differentiable manifold. For if  $A$  is a field of differential n-forms and  $\{x^a\}$  a local coordinate system in  $U \subset X$ , then with  $A$  we can associate a one-component geometric object  $E$  by

$$E: (\rho, \{x^a\}) \rightarrow E(x^a)$$

$$A = E(x^a) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

Using the results of section 2.8 we easily see that  $E$  is a scalar density, and the integral of  $A$  over any region is defined to be the integral of  $E$  over that region as defined above.

The exposition of elements of differential geometry given in this chapter follows rather closely that of K. Nomizu, Lie Groups and Differential Geometry, Publ. Math. Soc. Japan, No. 2, 1956.

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## 5. THEORIES OF SPACE, TIME AND GRAVITATION

### 5.1. SPACE-TIME AS A DIFFERENTIABLE MANIFOLD

The most fundamental principle of physics, common to all physical theories so far put forward, is that space and time can be represented by a 4-dimensional differentiable manifold. This is often considered as so obvious that it is hardly worth mentioning or analyzing, but we consider it to be worthy of some attention, as did Einstein who, in his exposition of the theory of relativity, devoted considerable attention to the problem: why do we consider space and time to be a continuum (by which he meant what is nowadays called a differentiable manifold)?

If one ignores quantum phenomena, including the atom-structure of matter, and assumes matter to be infinitely divisible so that there is no inherent lower limit on the extension of bodies, we shall show that it is plausible that a differentiable manifold is an appropriate model of space-time. One can imagine that there is in space a very large number of very small clocks, i.e. small bodies with a mechanism that can be used to indicate time. They need not be 'good' clocks in any sense of the word 'good.' All that is required is that they associate a number with every instant of time, and that two instants of time are never given the same number by the same clock. Further, each clock has three identification numbers engraved on it, such that no two clocks bear the same three numbers. The fact that three numbers are both necessary and sufficient to distinguish the clocks is the meaning of space being 3-dimensional, and is to be taken as an experimental fact. Time is only one-dimensional, as the clocks need indicate only one number completely to specify an instant of time. Together, we see that we need 4 numbers to specify a point in space and time, and so space-time is 4-dimensional.

If the identification numbers and the time readings are such that the neighboring clocks in space bear numbers differing very little from one another and indicate times differing very little, then they can be considered as defining a coordinate system in a differentiable manifold representation of space-time. Such a coordinate system can be used to analyze all physical phenomena in terms of coincidences of

events with these clocks. For example, the motion of a particle may be described by saying that the particle coincides with clock (1,3,-5) at time 6 by that clock, with clock (2,3,-6) at time 7 by that clock, with clock (3,3,-6) at time 8 ... , filling in as many events along the path of the clock as is desired. But to carry through this procedure it must be possible to associate clocks with as many points of space as one desires, which requires that we be able to make the clocks arbitrarily small, or we would not be able to find room for them at all; and furthermore it must be possible to read the dial of each clock with arbitrarily high accuracy. Perhaps we should point out that such a system of clocks has nothing to do with the metrical properties of space-time; it is necessary in order for space and time to be considered even as a differentiable manifold.

The situation becomes essentially more complicated when one takes into account the atomic, quantum nature of matter. It does not seem to be possible to have clocks whose linear dimensions are smaller than, say, 10<sup>-13</sup> cm. Moreover, such small clocks would be subject to quantum laws, so that if one knows with great accuracy where one such clock is at a given instant of time, at a later time its position will be very indeterminate. Even worse, such elementary clocks are, by quantum principles, indistinguishable, so we can neither engrave them with numbers to identify one from another, nor provide them with dials to indicate the time, etc! In fact, they would be completely useless for our purposes. This shows that one really has to use macroscopic clocks, composed of very large numbers of atoms, and then we cannot pack enough such clocks together to form a very useful coordinate system. And if it is not possible to construct, even in principle, a coordinate system in space and time, does it make sense to assume that space and time can be represented by a differentiable manifold? The question seems to be open, but one should not assume it as obvious that it is so.

Many people have expressed doubts along the above lines, and some have tried to construct physical theories from new assumptions about the structure of space and time, but up to now, no such attempt has met with much success. The first idea that suggests itself is to represent space and time by a regular lattice of points like that of atoms in a crystal, but one can easily see that this will not work. The symmetry group of the lattice would have to approximate to the Lorentz group, and one cannot find a lattice which would admit a symmetry transformation corresponding to a Lorentz transformation with very small velocity. A recent discussion containing arguments against the continuity of space and time is given by Bohm.

It has been said that the modern approaches to elementary particle theory based on S-matrix theory and related theories--such as dispersion relation theories--are not based on the assumption of a space-time continuum. But actually this statement is unfounded, as they require invariance under the inhomogeneous Lorentz group, from which we can construct the usual Minkowski space-time of special relativity.

From now on we shall always assume that space-time can be represented by a 4-dimensional differentiable manifold. This is why the differentiable manifold concept was defined with care and discussed in detail in the preceding chapter. Any changes in this assumption would result in a very profound revolution in physics.

## 5.2. THE AFFINE CONNECTION IN PHYSICS

There is another fundamental principle of physics that is common to all physical theories so far put forward. This is that the differentiable manifold of space and time is endowed with an affine connection whose geodesics form a privileged set of world-lines in space-time. The particular affine connection depends on the theory under consideration and may also depend on the particular solution of the theory we are considering, but the existence of an affine connection is common to all theories. It is necessary in order that the fundamental laws of physics can be expressed in the form of differential equations, which is certainly true of all physical theories.

Since one must be able to determine by physical experiment every mathematical construct introduced in a physical theory, we must look for a method of physically determining the affine connection. The symmetric part of an affine connection is determined when the totality of geodesics in the manifold is known. Not every family of curves in a differentiable manifold can be interpreted as the geodesics of an affine connection, so we must look for a privileged set of world-lines in space-time which can be so interpreted. Of course, this is not the only way of determining physically the symmetric part of the affine connection, but it is the most natural way. If a theory postulates a non-symmetric affine connection, the antisymmetric part, i.e. the torsion tensor, must be separately determined.

So we see that the first question we should ask of a physical theory is: What physically determined privileged class of world-lines is to be interpreted as the totality of geodesics of the affine connection of the space-time manifold? To this question each theory gives its own answer, and to proceed further we must specialize to a particular theory. This we shall now do.