

**TIME AND INTERPRETATIONS OF QUANTUM GRAVITY**

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## CONTENTS

1. Constraints and Dynamics . . . . .	1
2. Problems of Time in Quantum Gravity . . . . .	6
3. The Role of Models . . . . .	8
4. The Problem of Functional Evolution . . . . .	14
5. Interpretations of Quantum Gravity . . . . .	18
6. Internal Schrödinger Interpretation . . . . .	21
7. Matter Clocks and Reference Fluids . . . . .	32
8. Unspecified Cosmological Constant and Cosmological Time . . . . .	39
9. Klein-Gordon Interpretation . . . . .	42
10. Semiclassical Interpretation . . . . .	47
11. Third Quantization . . . . .	62
12. Naive Schrödinger Interpretation . . . . .	66
13. Conditional Probability Interpretation . . . . .	69
14. Sum-Over-Histories Interpretation . . . . .	76
15. Frozen Time Formalism and Evolving Constants of the Motion . . . . .	82
16. Summary . . . . .	88
17. Acknowledgements . . . . .	91
18. References . . . . .	91

## TIME AND INTERPRETATIONS OF QUANTUM GRAVITY

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### ABSTRACT

In canonical quantization of gravity, the state functional does not seem to depend on time. This hampers the physical interpretation of quantum gravity. I critically examine ten major attempts to circumvent this problem and discuss their shortcomings.

Mein sind die Jahre nicht, die mir die Zeit genommen;  
Mein sind die Jahre nicht, die etwa möchten kommen;  
Der Augenblick ist mein, und nehm ich den in acht,  
So ist der mein, der Jahr und Ewigkeit gemacht.

*Betrachtung der Zeit*  
*Andreas Gryphius 1616-1664*

### 1. Constraints and Dynamics

In the midst of the Thirty Years War, Andreas Gryphius contemplated the uncertainties of his time in an epigram about Time itself:<sup>1</sup>

Those years are not my years that were devoured by time,  
Those years are not my years that fate may deem be mine;  
This instant, though, is mine, and if I hold it dear,  
Then He is mine who made the endless train of years.

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I can hardly think of a better summary of the aims of canonical geometrodynamics.

Of course, to predict the future and reconstruct the past from the present data has been the main preoccupation of physicists long before Einstein's theory of gravitation was cast into canonical form. I do not want to read Gryphius' words in this trivial way. Canonical gravity is more than a complicated implementation of the Laplace program: When predicting the future and reconstructing the past, it relies on dynamical laws which are derived from the laws ruling the data at a single instant. While in every dynamical theory the future follows from the present, in relativity the laws according to which this happens also follow from the laws of the present.

To understand the relationship between the dynamical laws and the laws of an instant requires some explanation. The dynamical laws were introduced into physics by Newton; the first law of an instant was discovered by Gauss. In electrodynamics, the electric field cannot be arbitrarily prescribed, but it is subject to the Gauss law: In vacuo, the divergence of  $E$  must vanish. This is not a dynamical law, but a limitation on the instantaneous distribution of the electromagnetic data. The dynamical Maxwell equations preserve the Gauss law, but they do not follow from it.

In canonical geometrodynamics, the instantaneous state of a vacuum Einstein spacetime is described by the intrinsic geometry  $g_{ab}(x)$  and the extrinsic curvature  $p^{ab}(x)$  of a spacelike hypersurface  $\Sigma \rightarrow \mathcal{M}$ . These instantaneous data are not arbitrary; at any point  $x \in \Sigma$ , four combinations of the data,

$$H_a(x) = -2D_b p_a^b(x) \quad (1.1)$$

and

$$H(x) = G_{abcd}(x;g) p^{ab}(x) p^{cd}(x) - |g|^{1/2} R(x;g), \quad (1.2)$$

$$G_{abcd} = \frac{1}{2} |g|^{-1/2} (g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}), \quad |g| := \det(g_{ab})$$

called the supermomentum and the super-Hamiltonian, are constrained to vanish:

$$H_a(x; g,p] = 0 \quad (1.3)$$

and

$$H(x; g,p] = 0 \quad (1.4)$$

(The notation  $H(x; g,p]$  emphasizes that  $H$  is a function of  $x$  but a functional of  $g$  and  $p$ .) These are the laws of an instant in canonical gravity. The divergence

law (1.1), (1.3) is analogous to the Gauss law; the super-Hamiltonian constraint (1.2), (1.4) has no counterpart in electrodynamics. It is this constraint that ultimately yields the dynamics of geometry.

Any reference to the hypersurface  $\Sigma \rightarrow \mathcal{M}$  which carries the geometric data  $g_{ab}(x)$ ,  $p^{ab}(x)$  is conspicuously absent in the constraints (1.1) - (1.4). The hypersurface  $\Sigma \rightarrow \mathcal{M}$  represents an instant of time; the fact that it drops out of the constraints (1.1) - (1.4) underlies the problem of time in quantum gravity.

One can surmise that the geometric data  $g_{ab}(x)$ ,  $p^{ab}(x)$  themselves specify the hypersurface which carries them (Baierlein, Sharp and Wheeler<sup>2</sup>). One suspects there is a canonical transformation<sup>3</sup>

$$g_{ab}(x), p^{ab}(x) \rightarrow X^A(x), P_A(x), \phi^r(x), \pi_r(x) \quad (1.5)$$

which separates the four *embedding variables*  $X^A(x)$ ,  $A = 0, 1, 2, 3$  that specify the hypersurface from the two *true gravitational degrees of freedom*  $\phi^r(x)$ ,  $r = 1, 2$ . One could then solve the constraints (1.1) - (1.4) with respect to the *embedding momenta*  $P_A(x)$  and, after doing so, replace them by an equivalent set of constraints

$$H_A(x) := P_A(x) + h_A(x; X, \phi, \pi) = 0 . \quad (1.6)$$

The expressions  $h_A(x)$  represent the energy density and the energy flux carried by the gravitational variables  $\phi$  and  $\pi$  through the hypersurface  $X^A(x)$ .

This sets the stage for deriving the dynamical laws from the constraints (1.1) - (1.4), or (1.6), i.e., from the laws of an instant. Among several alternative routes, the one which is closest to quantum gravity relies on the Hamilton-Jacobi theory. One postulates that the momenta  $p^{ab}(x)$  (or  $P_A(x)$ ,  $\pi_r(x)$ ) are generated by a Jacobi principal functional  $S[g]$  (or  $S[X, \phi]$ ):

$$p^{ab}(x) = \delta S[g] / \delta g_{ab}(x) , \quad or \quad (1.7)$$

$$P_A(x) = \delta S[X, \phi] / \delta X^A(x), \quad \pi_r(x) = \delta S[X, \phi] / \delta \phi^r(x).$$

By substituting the expressions (1.7) into the constraints (1.1) - (1.4) or (1.6),

$$- 2D_b \frac{\delta S[g]}{\delta g_{ab}(x)} = 0 . \quad (1.8)$$

$$H\left(x; g, \frac{\delta S[g]}{\delta g}\right) = 0 \quad , \quad (1.9)$$

or

$$\frac{\delta S[X, \phi]}{\delta X^A(x)} + h_A\left(x; X, \frac{\delta S[X, \phi]}{\delta \phi}\right) = 0 \quad , \quad (1.10)$$

one obtains variational differential equations for the Jacobi principal functional  $S[g]$  or  $S[X, \phi]$ . These are the Hamilton-Jacobi equations in geometrodynamics.<sup>4,5,3</sup> A particular solution of Eqs. (1.8) - (1.9) (or Eq. (1.10)) generates a family of solutions of the Einstein equations.<sup>6</sup> A complete solution generates a canonical transformation which yields all the solutions of the Einstein equations.<sup>7,5</sup> In this way, the dynamical laws according to which geometry evolves follow from the constraints, i.e., from the laws of an instant.

The original constraints (1.1) - (1.4) lead to Eqs. (1.8) and (1.9). Equation (1.8) ensures that the Jacobi principal functional remains unchanged under  $\text{Diff } \Sigma$ , i.e., that it does not depend on the metric  $g_{ab}(x)$ , but only on the geometry  $g(x)$ . Equation (1.9) is then the proper Hamilton-Jacobi equation for  $S[g]$ . In it, the dynamical degrees of freedom are not separated from the time (from the embedding variables) in which they evolve. This happens only in Eq. (1.10) which has the form of a first-order functional differential equation in  $X^A(x)$ .

- **Global Problem of Time.** Even at the classical level, the extraction of dynamics from the instantaneous constraints poses some problems. First, the separation of time (of the embedding variables) from the dynamical degrees of freedom may be globally impossible. Such a situation occurs already in particle dynamics. In parametrized Newtonian dynamics,<sup>8</sup> the constraints are automatically given in the form (1.6) with respect to the absolute time  $T$ :

$$H := P_T + h(T; Q, P) = 0 \quad (1.11)$$

The linearity of the constraint (1.11) in  $P_T$  implies that the absolute time  $T$  always increases along any dynamical trajectory and hence every trajectory intersects a hypersurface of constant  $T$  once and only once. However, some other simple finite-dimensional constrained dynamical systems may admit no time function with this property.<sup>9-12</sup> This indicates that the super-Hamiltonian constraint of such systems cannot be globally replaced by an equivalent constraint (1.11). The analysis of homogeneous cosmological models reveals that geometrodynamics meets similar problems.<sup>13</sup> So far, no one seems to have analyzed in detail how

serious and generic such problems are in full geometrodynamics (1.1) - (1.4). It is, however, quite likely that there does not exist any canonical transformation (1.5) such that the original constraints (1.3) - (1.4) are fully equivalent to the new constraints (1.6). If so, the Hamilton-Jacobi equation (1.10) cannot generate all Einstein spacetimes. I shall call this purely classical problem *the global problem of time*.

- **The Sandwich Problem.** The Hamilton-Jacobi equation (1.9) has a related difficulty. The choice of the *metric representation*  $S[g]$  presupposes that the evolution of the gravitational field is described by a canonical transformation whose generating function (the Hamilton principal function<sup>14,15</sup>) is a functional of the initial and final metric:  $S[g_{FIN}|g_{IN}]$ . It is questionable to what extent the assignment of the *intrinsic metric* on the initial and final hypersurfaces uniquely determines a spacetime geometry in the sandwich between them. This constitutes the so called *sandwich problem*.<sup>2,16-25</sup> In general, one would like to know what choice of canonical variables, if any, as arguments of the Hamilton principal function, yields a unique spacetime geometry in the sandwich. This question leads back to investigating canonical transformations like those given by Eq. (1.5).

- **Independence of the Dynamic Evolution on the Foliation.** The last question one needs to address is whether the dynamics generated by the Jacobi function (1.8) - (1.9), or (1.10), preserves the constraints (1.3) - (1.4), or (1.6), which lead to this dynamics. It is well known<sup>26</sup> this is so if the constraints are first class, i.e., if the Poisson brackets between any two of them vanish on the constraint hypersurface. The constraints (1.3) and (1.4) are first class,

$$\{H(x), H(x')\} \approx 0 \quad , \quad \text{etc.}, \quad (1.12)$$

by virtue of the so called Dirac algebra.<sup>27</sup> One can show then by a general argument that if the constraints (1.3) - (1.4) can be brought into the new form (1.6) by a canonical transformation (1.5), the Poisson brackets of the new constraints (1.6) must actually vanish strongly rather than weakly:<sup>28</sup>

$$\{H_A(x), H_B(x')\} = 0 \quad . \quad (1.13)$$

Equations (1.12) or (1.13) ensure that the evolution of the Jacobi principal functional by the functional Hamilton-Jacobi equation (1.8) - (1.9), or (1.10), from an initial hypersurface IN in a final hypersurface FIN does not depend on the foliation connecting IN with FIN. The functional nature of the time variable thus presents no problem in the classical theory.

## 2. Problems of Time in Quantum Gravity

The quantization of a system whose dynamics is completely described by first-class constraints can be accomplished by the Dirac constraint quantization.<sup>26</sup> First, the fundamental variables like  $g_{ab}(x)$ ,  $p^{ab}(x)$  or  $X^A(x)$ ,  $\phi^r(x)$ ,  $P_A(x)$ ,  $\pi_r(x)$  are turned into operators  $\hat{g}_{ab}(x)$ ,  $\hat{p}^{ab}(x)$  or  $\hat{X}^A(x)$ ,  $\hat{\phi}^r(x)$ ,  $\hat{P}_A(x)$ ,  $\hat{\pi}_r(x)$  which satisfy the fundamental commutation relations. By substituting these operators into Eqs. (1.1) - (1.2), or (1.6), one turns the constraint functionals into operators

$$\hat{H}_a(x) = -2 \hat{D}_a \hat{p}^b(x) \quad (2.1)$$

$$\hat{H}(x) = " \hat{p}^{ab}(x) G_{abcd}(x; \hat{g}) \hat{p}^{cd}(x) - |\hat{g}|^{1/2} R(x; \hat{g}) " \quad (2.2)$$

or

$$\hat{H}_A(x) = \hat{P}_A(x) + h_A(x; \hat{X}, \hat{\phi}, \hat{\pi}) \quad (2.3)$$

The quotation marks in Eq. (2.2) are a reminder that the factor ordering is ambiguous. The constraints are then imposed as limitations

$$H_a(x; \hat{g}, \hat{p}) \Psi[g] = 0 \quad (2.4)$$

$$H(x; \hat{g}, \hat{p}) \Psi[g] = 0 \quad (2.5)$$

or

$$H_A(x; \hat{X}, \hat{P}, \hat{\phi}, \hat{\pi}) \Psi[X, \phi] = 0 \quad (2.6)$$

on the physical states  $\Psi \in \mathcal{S}_0$  of the system.

Equation (2.4) ensures that the physical states depend only on the three-geometry  $\mathbf{g}(x)$ , not on the metric  $g_{ab}(x)$  which represents  $\mathbf{g}(x)$ :  $\Psi[g] = \Psi[\mathbf{g}]$ .<sup>29-30</sup> Equation (2.5) is a second-order functional differential equation for  $\Psi[g]$  called the *Wheeler-DeWitt equation*.<sup>18,31-32</sup> On the other hand, the constraint (2.6) has the form of a functional Schrödinger equation<sup>33</sup>

$$i \frac{\delta \Psi[X, \phi]}{\delta X^A(x)} = h_A(x; X, \hat{\phi}, \hat{\pi}) \Psi[X, \phi] \quad (2.7)$$

in the hypertime variables  $X^A(x)$ .

In my schematic discussion of the Dirac constraint quantization, I swept under the rug a heap of technical problems. These are carefully spelled out, e.g., by Isham.<sup>34</sup> I must, however, pinpoint those technical and conceptual problems



which constitute the problem of time in quantum gravity. Broadly speaking, there are three places in which things can go wrong in the transition from the classical to the quantum theory:

- **The Problem of Functional Evolution.** The involution conditions (1.12) or (1.13) do not need to be valid for the commutators of the constraint operators. Thus, instead of Eq. (1.13) one can get

$$[\hat{H}_A(x), \hat{H}_B(x')] \neq 0 \quad (2.8)$$

and instead of Eq. (1.12)

$$[\hat{H}(x), \hat{H}(x')] \Psi[g] \neq 0, \text{ etc., for a } \Psi \in \mathcal{F}_0 \quad (2.9)$$

If so, the functional equations (2.6), or (2.4) - (2.5), are inconsistent or, at best, admit only special solutions. Equations (2.8) and (2.9) mean that the evolution of state from an initial hypersurface IN to a final hypersurface FIN can depend on the foliation which connects FIN with IN. In other words, when one starts with the same initial state  $\Psi_{IN}$  on the initial hypersurface, and develops it to the final hypersurface along two different routes, one obtains two different states  $\Psi_{1FIN} \neq \Psi_{2FIN}$  on the final hypersurface. Such a situation certainly violates what one would expect of a relativistic theory. I shall call this problem *the problem of functional evolution*.

- **The Multiple Choice Problem.** The Schrödinger equation (2.7) based on one choice of the time variable  $X^A(x)$  may give a different quantum theory than the Schrödinger equation based on another choice of the time variable. I shall call this problem *the multiple choice problem*.

The multiple choice problem is quite distinct from the problem of functional evolution; the latter arises even if one sticks to a single choice of time. It is also distinct from the global problem of time in classical geometrodynamics. The multiple choice problem is one of an embarrassment of riches: out of many inequivalent options, one does not know which one to select. The global problem of time is an embarrassment of poverty: one really does not have any choice at all.

- **The Hilbert Space Problem.** The multiple choice problem stems from an attempt to separate time from the dynamical degrees of freedom. What happens when one sticks to the Wheeler-DeWitt equation in the metric representation? While the Schrödinger equation automatically brings in a conserved inner product (conserved in the selected time variable) and the

possibility to construct, at a given time  $X^A(x)$ , meaningful observables  $F[X, \hat{\phi}, \hat{\pi}]$  from the dynamical degrees of freedom, nothing similar is granted by the Wheeler-DeWitt equation. As a second-order functional differential equation, the Wheeler-DeWitt equation presents familiar problems when one tries to turn the space of its solutions into a Hilbert space. One thus trades the multiple choice problem for a *Hilbert space problem*.

The problem of functional evolution, the multiple choice problem, and the Hilbert space problem are the three major classes of problems which quantum geometrodynamics encounters because classical geometrodynamics does not seem to possess a natural time variable, while standard quantum theory relies quite heavily on a preferred time. Under these three headings, more specific problems will start emerging as I shall examine the proposals for solving or circumventing these major problems of time.

### 3. The Role of Models

Quantum gravity is an ill-defined and little understood system. It is no wonder that most people who tackled its time problem tried to elucidate it on simpler models. Each of these models shares with Einstein's theory some relevant feature which gives rise to a time problem. However, none of them shares with Einstein's theory all the relevant features. Therefore, even if one knows how to solve the time problem of the model, the time problem in the full Einstein theory remains unsolved.

I shall arrange eleven widely explored models in a hierarchical order, starting with those which closely resemble Einstein's theory, and then moving further and further away from it. The first four models are field theoretical systems, the remaining seven are finite-dimensional.

▪ **Midisuperspace Models.** These provide a canonical description of Einstein spacetimes with a given group of isometries. Symmetry removes infinitely many degrees of freedom of the gravitational field, but there are still infinitely many degrees of freedom left.

A prototype of a midisuperspace model is the cylindrical gravitational wave (Kuchař<sup>35</sup>). Cylindrical symmetry restricts the spatial line element to the form

$$d\sigma^2 = e^{\gamma-\psi} dr^2 + R^2 e^{-\psi} d\varphi^2 + e^\psi dz^2 \quad (3.1)$$

in which it is characterized by three functions  $\psi, \gamma$  and  $R \geq 0$  of a radial coordinate  $r$ . Similarly, the extrinsic curvature of an arbitrary hypersurface which

respects the cylindrical symmetry (i.e, which contains its two commuting Killing vectors) is completely characterized by the three conjugate momenta  $\pi_\psi$ ,  $p_\gamma$ , and  $p_R$ . When expressed in terms of these canonical variables, the super-Hamiltonian (1.2) assumes the form

$$H(r) = e^{\frac{1}{2}(\psi-\gamma)} \left( -p_\gamma p_R + \frac{1}{2} R^{-1} \pi_\psi^2 + \frac{1}{2} R \psi'^2 + 2R'' - \gamma'R' \right) . \quad (3.2)$$

Only the radial component  $H_1$  of the supermomentum is different from zero,

$$H_1(r) = -2p_\gamma' + \gamma'p_\gamma + R'p_R + \psi'\pi_\psi . \quad (3.3)$$

In spite of these simplifications, the midisuperspace constraints (3.2) and (3.3) are still fairly complicated functionals of the canonical variables  $\psi$ ,  $\gamma$ ,  $R$  and  $\pi_\psi$ ,  $p_\gamma$ ,  $p_R$ .

Other widely studied midisuperspace models are Gowdy cosmologies (Gowdy,<sup>36</sup> Misner,<sup>37</sup> Berger,<sup>38-40</sup> and Husain<sup>41</sup>).

- **Strings (Membranes).** Canonical analysis of these systems<sup>42-44</sup> leads to a system of constraints whose structure closely resembles that of Einstein's theory of gravitation. In particular, one obtains the standard  $\text{Diff}\Sigma$  (supermomentum) constraint and a quadratic super-Hamiltonian constraint, which close according to the Dirac algebra.

- **Parametrized Fields.** A field theory on a given Riemannian background can be made covariant under  $\text{Diff}\mathcal{M}$  by adjoining to the field variables the embedding variables specifying spacelike hypersurfaces in  $\mathcal{M}$ .<sup>26,45-47</sup> Canonical analysis of a parametrized field theory leads directly to a system of constraints which has the form (1.6), where  $\phi^f$  are the field variables,  $\pi_f$  their conjugate momenta, and  $X^A$  the embedding variables. The projection of these constraints tangential and normal to the hypersurface leads to the supermomentum  $H_a(x; X, P, \phi, \pi]$  and super-Hamiltonian  $H(x; X, P, \phi, \pi]$ , which again satisfy the Dirac algebra.

- **Inhomogeneous Perturbations of Homogeneous Cosmologies.** This class of models stands at the border between infinitely-dimensional and finite-dimensional systems. One takes a homogeneous cosmological model (see the next paragraph) and introduces infinitely many inhomogeneous modes by allowing its linear perturbations.<sup>48</sup>

▪ **Minisuperspace Models (Homogeneous Cosmologies).** As in the midisuperspace models, infinitely many degrees of freedom are frozen by symmetries. The reduction is so drastic that only a finite number of degrees of freedom is left. The models represent homogeneous (but in general anisotropic) cosmologies, in particular the various Bianchi-type vacuum spacetimes.<sup>32,49-52</sup> The requirement of homogeneity limits the allowed hypersurfaces to the leaves of a privileged foliation, which is labeled by a single time variable. A good example of a minisuperspace model is the Bianchi type IX *mixmaster universe*.<sup>49,53-55</sup> Its homogeneity group is  $SO(3)$ . One can parametrize the hypersurfaces of homogeneity by the Euler angle coordinates and characterize the spatial metric uniquely by three real parameters,  $\Omega$  and  $\beta_{\pm}$ . The first parameter,  $\Omega$ , is related to the volume of  $\Sigma$ :

$$\Omega = \ln \int_{\Sigma} d^3x |g(x)|^{1/2} . \quad (3.4)$$

The remaining two parameters,  $\beta_{\pm}$ , describe the anisotropy of  $\Sigma$ .

The extrinsic curvature of a hypersurface of homogeneity is similarly determined by the three momenta,  $p_{\Omega}$  and  $p_{\pm}$ , canonically conjugate to  $\Omega$  and  $\beta_{\pm}$ . Due to the symmetry of the model, the supermomentum constraints are again identically satisfied, while the super-Hamiltonian constraints reduce to a single constraint

$$H := -p_{\Omega}^2 + p_+^2 + p_-^2 + e^{-4\Omega} (V(\beta_+, \beta_-) - 1) = 0 \quad (3.5)$$

for the minisuperspace variables  $\Omega, \beta_{\pm}; p_{\Omega}, p_{\pm}$ . The potential  $V(\beta_+, \beta_-)$  is a combination of exponential terms; it vanishes at the origin  $\beta_+ = 0 = \beta_-$ , and it is positive outside the origin.

▪ **2+1 Gravity.** A three-dimensional Einstein spacetime is necessarily flat.<sup>56</sup> Thus, 2+1 gravity lacks field degrees of freedom. However, there is still a finite number of variables characterizing the topology of  $\Sigma$ .<sup>57-58</sup> The problem of time in the 2+1 gravity was discussed by Moncrief,<sup>59</sup> Hosoya and Nakao,<sup>60</sup> and Carlip.<sup>61,62</sup>

▪ **A Relativistic Particle.** A relativistic particle of rest mass  $M$  moving in a Riemannian spacetime  $(\mathcal{M}, G)$  is a dynamical system described by a single super-Hamiltonian

$$H := \frac{1}{2M} G^{AB}(Q)P_A P_B + \frac{1}{2}MV(Q) = 0 . \quad (3.6)$$

(I have allowed for the possibility that the mass of the particle is influenced by a

potential field  $V(Q)$ , as in Nordström's theory of gravitation<sup>63-65</sup>). By comparing Eq. (3.6) with Eq. (1.2), one sees that geometrodynamics is analogous to a relativistic particle, with the intrinsic geometry  $g_{ab}(x)$  playing the role of the spacetime position  $Q^A$ , DeWitt's supermetric  $G_{abcd}(x; g)$  the role of the background metric  $G^{AB}(Q)$ , and the scalar curvature  $-|g|^{1/2}R(x; g)$  the role of the potential  $V(Q)$ . By enlarging the spacetime by gauge degrees of freedom and cutting these down by additional gauge constraints, one can also mimick the supermomentum constraints (1.1).<sup>66</sup>

▪ **A Parametrized Newtonian Particle.** A Newtonian particle can be parametrized by labeling its trajectory by an arbitrary parameter and adjoining the Newtonian time to the position  $Q^a$ . Its motion is generated by the constraint (1.11). The constraint (1.11) can be rewritten in a geometric form in terms of the metric and affine structures of the Newtonian spacetime.<sup>67,68</sup> It resembles then the constraint (3.6) for a relativistic particle.

▪ **The BB Model.** A system of Newtonian particles  $K=1, \dots, N$  interacting by central forces can be parametrized by labeling the dynamical trajectories by an arbitrary parameter and adjoining the Newtonian time to the position variables  $Q_{(K)}^a$ . One can then constrain the system by the requirements that its total energy  $H$ , total momentum  $P$  and total angular momentum  $J$  vanish:

$$H = \sum_{K=1}^N \frac{1}{2M_{(K)}} \delta^{ab} P_{(K)a} P_{(K)b} + \sum_{K \neq L} \frac{1}{2} W(|Q_{(K)} - Q_{(L)}|), \quad (3.7)$$

$$P = \sum_{K=1}^N P_{(K)}, \quad J = \sum_{K=1}^N Q_{(K)} \times P_{(K)}. \quad (3.8)$$

This model was introduced by Barbour and Bertotti<sup>69-73</sup> to implement Mach's ideas. By comparing Eq. (3.7) with (1.2) and (3.8) with (1.1), one sees the analogy between the BB model and canonical geometrodynamics.

▪ **Two Particles in a Stationary State.** The first particle moves in a curved space with the metric  $G^{AB}(Q)$  and an external potential  $V(Q)$ . The second particle moves in a curved space with the metric  $h^{kl}(Q; q)$  and the potential  $v(Q; q)$ . An interaction of the two particles is brought in by a dependence of the metric  $h^{kl}$  and the potential  $v$  on the coordinates of the first particle. One constrains the motion of the system by requiring that it takes place with a given energy  $E$ :

$$\begin{aligned}
 H := & \frac{1}{2M} G^{AB}(Q)P_A P_B + \frac{1}{2} MV(Q) \\
 & + \frac{1}{2m} h^{kl}(Q;q)p_k p_l + \frac{1}{2}mv(Q;q) - E = 0 .
 \end{aligned}
 \tag{3.9}$$

The model can be specialized in a number of ways. The particles can move in the same space, the space may be flat, or the particles need not interact with each other. The metrics may be positive definite or indefinite. This class of models plays a useful role when discussing semiclassical gravity and the role of internal clocks.

▪ **Two Harmonic (Antiharmonic) Oscillators.** One can build a super-Hamiltonian

$$H = \left( \frac{1}{2M} P^2 \pm \frac{1}{2} M \Omega^2 Q^2 \right) \pm \left( \frac{1}{2m} p^2 \pm \frac{1}{2} m \omega^2 q^2 \right)
 \tag{3.10}$$

from the Hamiltonians for two harmonic (antiharmonic) oscillators, and impose the constraint  $H = 0$ . This is a special case of the previous model. The super-Hamiltonian (3.10) was used by Hájíček to analyze the global problem of time.<sup>11,12</sup> It is also useful to illustrate "group quantization".<sup>34</sup>

One can clarify various aspects of the time problem on the listed models. Unfortunately, it is much too easy to forget those simplifying features which distinguish them from canonical gravity, and succumb to an illusion that time problems in quantum geometrodynamics can be solved in exactly the same way as the particular time problem of the model. To dispel such an illusion, let me discuss several typical differences between the models and Einstein's theory of gravitation.

• **Auxiliary Structures.** A model may possess symmetries (in superspace) or other geometric structures which geometrodynamics lacks. This happens for cylindrical waves, bosonic strings, 2+1 gravity, the parametrized nonrelativistic particle, the oscillator system; for parametrized fields, the relativistic particle, and strings and membranes in stationary spacetimes; and for special minisuperspace models. When generalizing to quantum gravity, one should be careful that the proposed solution to the time problem does not depend on such auxiliary structures.

A particular time problem may be absent in a model:

- **No Multiple Choice Problem.** The constraint system for a parameterized nonrelativistic particle or a parameterized relativistic field is directly given in the form (1.6). There is thus no multiple choice problem. For other systems (like cylindrical waves or bosonic strings) there exists a natural choice (1.5) of the embedding variables which yields the constraints (1.6).

- **No Functional Evolution Problem.** There is no problem of functional evolution for finite-dimensional models. It is thus easy to forget that such a problem exists in any field theory. In particular, it is dangerous to draw conclusions from minisuperspace models to full quantum geometrodynamics. Minisuperspace spacetimes possess a privileged foliation by leaves of homogeneity which does not exist in a generic spacetime.

- **Positivity Restrictions.** Two parallels which are often drawn, one between geometrodynamics and a relativistic particle, and another one between geometrodynamics and the BB model, lead to diametrically opposite insights about the nature of time in geometrodynamics. In the first parallel, the intrinsic metric is compared with the position  $Q^A$  of a relativistic particle in *spacetime*. In the second parallel, it is compared with the position  $Q^a$  of a non-relativistic particle in *space*. The first parallel leads one to believe that the intrinsic metric contains a purely *kinematical* time variable (like  $Q^A$  contains  $Q^0$ ). The second parallel leads one to believe that time in geometrodynamics must be measured by observing a selected dynamical degree of freedom, say,  $Q^2$ , of a *realistic clock*.

It is important to realize that each of these parallels may be misleading. In all three systems (the relativistic particle, the BB model, and geometrodynamics) the super-Hamiltonian  $H$  is the sum

$$H = H_{\text{KIN}} + H_{\text{POT}} \tag{3.11}$$

of two terms, the kinetic term  $H_{\text{KIN}}$ , and the potential term  $H_{\text{POT}}$ . The kinetic term  $H_{\text{KIN}}$  is quadratic in the momenta, while the potential term  $H_{\text{POT}}$  depends only on the configuration variables. However, the individual terms have quite different *positivity properties* in the three systems I am considering. For a relativistic particle, the kinetic term is indefinite while the potential term is positive definite:  $H_{\text{POT}} \geq 0$ . In the BB model, the kinetic term is positive definite,  $H_{\text{KIN}} \geq 0$ , while the potential term is indefinite. In geometrodynamics, both the kinetic and the potential terms are indefinite.

The positivity of  $H_{\text{POT}}$  in the dynamics of a relativistic particle allows one to cast the super-Hamiltonian (3.6) into a form analogous to Eq. (1.6), where  $h$  is the (square-root) Hamiltonian of a relativistic particle. The expression under the square root is positive definite and the corresponding operator can be defined

by spectral analysis. This highlights the kinematical nature of the time variable. In the BB model, the positivity of the kinetic term  $H_{\text{KIN}}$  leads to theorems (Pauli,<sup>74</sup> Hartle,<sup>75</sup> Wald and Unruh<sup>76</sup>) which limit the accuracy with which a dynamical degree of freedom  $Q^2$  of a realistic clock can serve as an ideal kinematical clock. The indefiniteness of  $H_{\text{KIN}}$  in geometrodynamics prevents one from converting the Wheeler-DeWitt equation (2.5) into the Schrödinger equation (2.6) by square-rooting. The indefiniteness of  $H_{\text{KIN}}$  casts doubt on the relevance of the theorems proved by Hartle and by Wald and Unruh: geometrodynamics may still possess a kinematical time variable.

One should try to avoid a common hubris which consists of solving a time problem for a model way down in the hierarchy, and jumping to the conclusion that the time problems of quantum gravity are removed by the same treatment.

#### 4. The Problem of Functional Evolution

While people's response to the other issues of time tends to be intimately intertwined with their favorite interpretation of quantum gravity, the problem of functional evolution stands sufficiently apart to be discussed before I turn to the specific interpretative schemes. The actual form, (2.8) or (2.9), which the problem takes depends, of course, on whether one works within an internal time framework, or commits oneself to the Wheeler-DeWitt equation.

Much more than other time problems, the problem of functional evolution critically depends on our ability to construct gravity as a consistent quantum *field* theory, in particular, on giving sense to the infinite collection of constraint operators (2.1) - (2.3), and to their commutation relations (2.8) - (2.9). There is actually very little one can say about this problem in full quantum gravity before one succeeds in constructing the full quantum gravity. The best one can do is to elucidate what the problem means on sufficiently simple (and hence almost trivial) infinitely-dimensional models.

In this spirit, I have formulated and solved the problem of functional evolution for the parametrized massless scalar field propagating on a (1+1)-dimensional flat cylindrical Minkowskian background.<sup>77-79</sup> There is a canonical transformation which casts the super-Hamiltonian and supermomentum constraints in the midi-phase-space of the cylindrical gravitational waves into the constraints for a parametrized scalar field on a fictitious (1+1)-dimensional Minkowskian background.<sup>35</sup> This links the problem of functional evolution in the parametrized field theory to the one in quantum gravity. The classical steps needed to approach the problem of functional evolution for cylindrical gravitational waves were taken by Torre.<sup>80</sup> Similarly, there is a canonical transformation which casts the super-Hamiltonian and supermomentum constraints of a bosonic string moving in a d-



dimensional target space into those of a parametrized theory of  $d - 2$  independent massless scalar fields propagating on a 2-dimensional Minkowskian background. This allows one to formulate and solve the problem of functional evolution for a bosonic string.<sup>44,81</sup>

▪ **Functional Evolution for a Parametrized Scalar Field.** I shall introduce the problem of functional evolution on the example of a parametrized scalar field. Let  $X^A = (T, X)$ ,  $T \in (-\infty, \infty)$  and  $Z \in (-\pi, \pi)$  be the Minkowskian coordinates on a Minkowskian cylinder whose radius was chosen as a unit of length, and  $X^A(x)$  is any spacelike hypersurface (in this case, a curve) on the cylinder. The constraints of the parametrized massless scalar field theory then take the form (1.6), where the many-fingered Hamiltonian

$$h_A(x; X, \phi, \pi) = T_{AB}(x; X, \phi, \pi) |g(x; X)|^{1/2} n^B(x; X) \quad (4.1)$$

is obtained by projecting the energy-momentum tensor  $T^{AB}$  of the field into the (densitized) normal  $|g|^{1/2} n^B$  to the embedding  $X^A(x)$  and expressing the result in terms of the canonical variables  $\phi(x)$ ,  $\pi(x)$ . These constraints have the vanishing Poisson brackets (1.13).

To quantize the theory, one must factor order the Hamiltonian (4.1). An obvious way of doing this is to adopt the normal ordering  $:\hat{h}_A(x; X, \hat{\phi}, \hat{\pi}):$  with respect to the (Heisenberg) creation and annihilation operators corresponding to the initial data  $\hat{\phi}_o(x)$ ,  $\hat{\pi}_o(x)$  on a flat slice (maximal circle) of the cylinder. The problem of functional evolution arises because the commutators of the energy-momentum tensor operators  $:\hat{T}_{AB}(X):$  acquire Schwinger terms<sup>82</sup> which, due to the construction (1.6) and (4.1) of the constraints, creep into the commutator algebra of the constraints as an anomaly,

$$\frac{1}{i} [ :\hat{H}_A(x): , :\hat{H}_B(x'): ] = - F_{AB}(x, x'; X) . \quad (4.2)$$

The detailed form of  $F$  is not important for the further discussion. What is important is that  $F$  depends only on the embedding  $X^A(x)$ , not on the field variables  $\hat{\phi}(x)$  and  $\hat{\pi}(x)$ .

The anomaly (4.2) makes the constraints (2.6) inconsistent: by applying Eq. (4.2) to a state  $\Psi$  which satisfies the constraints (2.6), one learns that the state functional must vanish. If the Dirac constraint quantization fails for such a simple system as a parametrized field on a flat background, how could one ever dare to apply it to quantum gravity?

The resolution of the problem requires an exploration of the structure of the anomaly as a functional of the embedding variables. The Jacobi identity for

the commutator (4.2) amounts to the statement that the anomaly is a closed 2-form in the space of embeddings. Indeed, the anomaly 2-form is exact, i.e., it is a functional exterior derivative of a potential  $A_A(x; X]$ . This potential has two parts: the standard Casimir energy-momentum flux which is due to the finite spatial size of the Minkowskian cylinder, and the interesting piece, which depends only on the extrinsic curvature of the embedding  $X^A(x)$ , i.e., on the bending of the hypersurface.

One can redefine the constraint operators  $:\hat{H}_A:$  by adding to them the anomaly potential:

$$\hat{H}_A(x) := :\hat{H}_A(x): + A_A(x; X]. \quad (4.3)$$

This is tantamount to a hypersurface-dependent reordering of the constraints. The anomaly potential exactly cancels the anomaly, so that the new constraint operators (4.3) commute with each other, Eq. (2.8). The Dirac constraint quantization based on the reordered constraints (4.3) solves the problem of functional evolution.

▪ **Possible Generalizations.** The same scheme helps to resolve the functional evolution problem in the Dirac constraint quantization of a bosonic string<sup>44,81</sup> and, I guess, in the quantum geometrodynamics of cylindrical gravitational waves. It should be possible to generalize these results to parametrized free field theories on a flat four-dimensional background, but this was actually never done. (One should somehow take care of a single infinite constant which in four dimensions multiplies the Schwinger terms.) It would then be worthwhile to spell out the difficulties one can expect to arise for parametrized free fields on a curved background, or for parametrized interacting fields on a Minkowskian background. In the first case, it is not clear how to factor order the Hamiltonian, in the second case, it is not clear if the right-hand side of Eq. (4.2) does not also depend on the dynamical variables. By analyzing such difficulties, one would gauge at least some of the problems of quantum gravity approached via the many-fingered time Schrödinger equation (2.7).

• **Functional Evolution Problem for the Wheeler-DeWitt Equation.** In the Wheeler-DeWitt approach to quantum gravity, the problem of functional evolution is encoded in Eq. (2.9). Its resolution is equivalent to solving the notorious factor-ordering problem: To factor order the constraints (2.1) – (2.2) so that the commutator (2.9) of two super-Hamiltonians does not engender more constraints.

To resolve the problem, one should find a factor ordering of the super-Hamiltonian and supermomentum constraints such that the commutator of the

super-Hamiltonians yields an expression in which the supermomentum acts on the state function first, followed by the quantum version of the structure functions of the Dirac algebra. It was noticed by Anderson<sup>83,84</sup> that this task cannot be achieved if one insists on representing the super-Hamiltonian and supermomentum by self-adjoint operators on  $\mathcal{F}$ . A solution to the factor-ordering problem was offered by Schwinger<sup>85</sup> and criticized by Dirac.<sup>86</sup> The best, and certainly the shortest, exposition of Schwinger's solution may be found in a footnote of the paper<sup>32</sup> by DeWitt; this was later rediscovered by Komar.<sup>87,88</sup> DeWitt himself made a rather sweeping proposal on how to remove the problem by letting any two field operators taken at the same point formally commute.<sup>32</sup> Ashtekar has expressed the Hamiltonian theory of gravity in a new set of fundamental canonical variables and proposed a simple factor ordering of the constraints which (disregarding the regularization difficulties) satisfy the consistency requirement.<sup>89</sup> Unfortunately, all these results are purely formal: Tsamis and Woodard<sup>90</sup>, and Friedman and Jack<sup>91,92</sup> have persuasively argued that by formal manipulations of the commutator one can obtain whatever result one wants. The problem of functional evolution in the Wheeler-DeWitt equation can be tackled only when the field-theoretical complications are honestly faced and the discussion is raised above the purely formal level.

The factor-ordering problem of the Wheeler-DeWitt equation can be divested of its field theoretical aspects by simulating it on finite-dimensional models, mainly on the relativistic particle with extra gauge degrees of freedom. The resolution of this model problem was given by Hájíček and Kuchař.<sup>66,93</sup>

▪ **Fixing the Foliation.** The problem of functional evolution is so formidable that one may be tempted to give up and accept a draconian solution: to fix the foliation. Instead of measuring the gravitational degrees of freedom  $\phi^f(x)$  and  $\pi_r(x)$  on an arbitrary hypersurface, one can ask what happens when one attempts to measure them only on the leaves of a constant internal time  $T(x; g, p]$  in the frame  $Z^a(x; q, p]$  :

$$T(x; g, p] = \tau = \text{const}, \quad Z^a(x; g, p] = x^a. \quad (4.4)$$

For the purpose of evolving the state functional  $\Psi[X, \phi]$  along the foliation (4.4), one can replace it by the reduced functional

$$\Psi(t; \phi] := \Psi[ T(x) = t, Z^a(x) = x^a, \phi(x) ] \quad (4.5)$$

which is evolved by an ordinary Schrödinger equation

$$i\partial_t \Psi(t; \phi] = h(t; \hat{\phi}, \hat{\pi}] \Psi(t; \phi] \quad (4.6)$$

with the true Hamiltonian

$$\hat{h} = \int d^3Z h_0[ T = t, Z^a = x^a; \hat{\phi}, \hat{\pi} ] . \quad (4.7)$$

It is easy to see that when the many-fingered time Schrödinger equation (2.7) is consistent, and  $\Psi[X, \phi]$  is its solution, the restriction (4.5) of that solution satisfies the ordinary Schrödinger equation (4.6). However, Eq. (4.6) - (4.7) for  $\Psi(t; \phi]$  can make sense even if the functional Schrödinger equation (2.7) is inconsistent, Eq. (2.8). There is no problem of functional evolution when there is only one foliation along which the evolution can take place.

The foliation-fixing equations (4.4) can be considered as additional constraints which, when adjoined to the original constraints (1.6), make the total system of constraints second class. When the second class constraints are eliminated at the classical level and one quantizes only the reduced system, one recovers the Schrödinger equation (4.6) - (4.7). This is how the foliation fixing is ordinarily described.<sup>94</sup>

- **Foliation Fixing Versus General Relativity.** The foliation fixing prevents one from asking what would happen if one attempted to measure the gravitational degrees of freedom on an arbitrary hypersurface. Such a solution to the problem of functional evolution amounts to conceding that one can quantize gravity only by giving up general relativity: to say that quantum gravity makes sense only when one fixes the foliation is essentially the same thing as saying that quantum gravity makes sense only in one coordinate system.

## 5. Interpretations of Quantum Gravity

Quite a number of different proposals were made over the years on how to interpret quantum gravity. There are basically three ways in which they tried to cope with the problem of time:

I. **Internal Time Framework.** Time is hidden among the canonical variables and it must be identified prior to quantization. The basic equation on which the interpretation is based is a Schrödinger equation, not the Wheeler-DeWitt equation. This class of interpretations is prone to the multiple choice problem.

II. **Wheeler-DeWitt Framework.** The constraints are imposed in the metric representation and they yield the Wheeler-DeWitt equation. One strives for a dynamical interpretation of the solutions of the Wheeler-DeWitt equation which would be insensitive to the identification of time from among the metric

variables. This class of interpretations is prone to the Hilbert space problem.

**III. Quantum Gravity Without Time.** This class of interpretations is again mostly (though not necessarily) based on the Wheeler-DeWitt equation. It stresses the idea that one does not need time to interpret quantum gravity or, indeed, quantum mechanics in general. Time may emerge in particular situations, but even if it does not, quantum states still allow a probabilistic interpretation.

I find this classification of interpretations useful, even if the boundaries of the three classes are not sharply defined. I shall discuss ten representative attempts at interpreting quantum gravity under these headings.

#### **I. Internal Time Framework.**

▪ **Internal Schrödinger Interpretation.** Time (in the form of the embedding variables) is separated from the dynamical degrees of freedom by a canonical transformation in the geometric phase space prior to quantization. The constraints are resolved for the embedding momenta and in this form imposed on the physical states. The probabilistic interpretation is based on the resulting Schrödinger equation.

▪ **Matter Clocks and Reference Fluids.** The standard of time is provided by a matter system coupled to geometry rather than by the geometry itself. The introduction of matter facilitates the handling of the constraints which yield the Schrödinger equation.

▪ **Unspecified Cosmological Constant.** The cosmological constant  $\lambda$  is considered as a (conserved) dynamical variable. The super-Hamiltonian constraint yields the Schrödinger equation in the "cosmological time" conjugate to  $\lambda$ .

#### **II. Wheeler-DeWitt Framework.**

▪ **Klein-Gordon Interpretation.** The Wheeler-DeWitt equation is considered as an infinitely-dimensional analogue of the Klein-Gordon equation for a relativistic particle. The statistical interpretation of the theory is based on the Klein-Gordon norm, which is expected to be positive on a suitably restricted subspace of solutions.

▪ **Semiclassical Interpretation.** It is contended that the Wheeler-DeWitt equation has a probabilistic interpretation only if the Universe is in a semiclassical state. In this case, the Wheeler-DeWitt equation is approximated by the

Schrödinger equation.

- **Third Quantization.** The problems presented by the indefinite inner product of the Klein-Gordon interpretation are faced by suggesting that the solutions of the Wheeler-DeWitt equation are to be turned into operators. This is analogous to subjecting the relativistic particle whose state is described by the Klein-Gordon equation to second quantization.

### III. Quantum Theory Without Time.

- **Naive Schrödinger Interpretation.** It is suggested that the square  $|\Psi[g]|^2$  of a solution  $\Psi[g]$  of the Wheeler-DeWitt equation is the probability density for finding a spacelike hypersurface which carries the geometry  $g$ .

- **Conditional Probability Interpretation.** This is an elaboration of the naive Schrödinger interpretation. Again, the space of solutions of the *Wheeler-DeWitt* equation is equipped with the *Schrödinger* norm. For any two projection operators  $\hat{A}$  and  $\hat{B}$ , one gives the conditional probability that the answer to  $\hat{A}$  is yes if the answer to  $\hat{B}$  is yes. In special cases,  $\hat{B}$  is supposed to define an instant of time at which  $\hat{A}$  is measured.

- **Sum-Over-Histories Interpretation.** Path integrals are claimed to make sense even outside the Hilbert space framework. They are supposed to provide an interpretation of quantum systems which do not necessarily carry with them the notion of time.

- **Frozen Time and Evolving Constants of the Motion.** It is maintained that observables in quantum gravity are those dynamical variables which commute with all the constraints, i.e., which are the constants of the motion. An attempt is made to explain how such observables can describe an evolving Universe.

I shall devote the rest of my study to a critical examination of these ten interpretations. I have reviewed the issue of time in quantum gravity on two previous occasions, at the second Oxford Symposium on Quantum Gravity<sup>33</sup> and at the Osgood Hill Meeting.<sup>95</sup> My present analysis is parallel to that given by Isham at the 1991 Schladming Winter School.<sup>96</sup> Alternative reviews of some of these interpretations were given by Unruh and Wald<sup>76</sup> and by Barbour and Smolin.<sup>97</sup>

## 6. Internal Schrödinger Interpretation

I shall now discuss those interpretations of quantum gravity which insist that time is to be identified from among the geometric canonical variables and that the classical constraints must be cast into a very special form before quantization. Let me sketch the sequence of steps by which this is done:

- I. The embedding variables  $X^\Lambda(x)$  are separated from the true gravitational degrees of freedom  $\phi^f(x)$  by a canonical transformation (1.5).
- II. The constraints (1.1) - (1.4) are solved with respect to the embedding momenta and then replaced by a classically equivalent set of constraints (1.6).
- III. Those new constraints are turned into operators (2.3) and imposed as limitations (2.6) on the physical states.

The outcome of this procedure is a functional Schrödinger equation (2.7).<sup>33,35,78,79,98</sup> Indeed, this is the justification of all the trouble. One knows pretty well how to interpret a state function(al) which satisfies a Schrödinger equation, while it is not at all clear how to interpret a state function(al) which satisfies the original form (2.4) - (2.5) of the quantum constraints.

Provided that the functional evolution problem is solved, the functional Schrödinger equation is internally consistent and one can find its solutions  $\Psi[X,\phi]$ . At least formally, the functional integral

$$\langle \Psi | \Psi \rangle := \int D\phi |\Psi[X,\phi]|^2 \quad (6.1)$$

does not depend on the embedding  $X^\Lambda(x)$ , i.e., on the internal many-fingered time.<sup>33</sup> The inner product (6.1) turns the space  $\mathcal{S}_0$  of solutions  $\Psi[X,\phi]$  of the Schrödinger equation (2.7) into a Hilbert space. The Hilbert space structure provides the customary statistical interpretation of the quantized system. In particular,

$$D\phi |\Psi[X,\phi]|^2 \quad (6.2)$$

is interpreted as the probability to find the true gravitational degrees of freedom  $\phi^f(x)$  lying in the cell  $D\phi$  on a hypersurface  $X^\Lambda = X^\Lambda(x)$  specified by the internal embedding variables. More generally, the inner product (6.1) allows one to construct meaningful *quantum observables*. Any functional

$$\hat{F} = F[X, \hat{\phi}, \hat{\pi}] \quad (6.3)$$

of the true gravitational variables  $\hat{\phi}(x), \hat{\pi}(x)$  and of the embedding  $X^A(x)$  which is self-adjoint under the inner product (6.1) is an observable. An observable does not need to be a *constant of the motion*, i.e., the operator  $\hat{F}$  does not need to commute with the constraints (2.3) on the space of solutions:

$$[\hat{F}, \hat{H}_A(x)] \Psi \neq 0 \text{ for } \Psi \in \mathcal{S}_0. \quad (6.4)$$

Nevertheless, the familiar rules of quantum theory allow one to ask and answer questions about the spectrum of  $\hat{F}$  and the probability that, in the state  $\Psi$ , the observable  $\hat{F}$  on the hypersurface  $X^A(x)$  assumes a value  $F$  allowed by its spectrum.

The basic question in this program is how to choose the internal time variable. Three broad classes of options have been explored:

- **Intrinsic Time**  $T(x; g)$ . One assumes that the time variable is constructed entirely out of the intrinsic metric  $g_{ab}$  carried by the hypersurface.
- **Extrinsic Time**  $T(x; g, p)$ . To identify the hypersurface, one needs to know not only its intrinsic metric, but also its extrinsic curvature, i.e., how the hypersurface is bent in the encompassing Einstein spacetime.
- **Matter Time**. Time is to be constructed, not from the geometric data, but from matter fields coupled to gravity. I discuss this possibility in the next section. For the time being, I will confine my attention to vacuum gravity.

Very little is known about how to choose an internal time and make sense of the resulting Schrödinger equation in full quantum gravity. Indeed, the only general scheme which was seriously contemplated was to foliate spacetime either by maximal hypersurfaces or by hypersurfaces of constant mean extrinsic curvature, and call these the hypersurfaces of constant time. The remaining proposals were made on models. I shall mention these first, and then return to the mean extrinsic curvature scheme.

- **Homogeneous Cosmologies**. A single real variable, rather than a function, is needed to label the leaves of the privileged foliation on which both the intrinsic metric and the extrinsic curvature are homogeneous. The early work on homogeneous models revealed that a useful internal time variable labeling the leaves is the logarithm (3.4) of their total volume.<sup>53,54</sup> The choice (3.4) is a



primary example of an intrinsic time. Other choices, including the mean extrinsic curvature of the leaves, were also explored.<sup>13,96</sup>

To bring the super-Hamiltonian constraint of the model (like Eq. (3.5) for the mixmaster universe) to the Schrödinger form, one must solve a quadratic equation for  $p_\alpha$ .

■ **Linearized Gravity.** The advantages of an extrinsic time as a tool for casting geometrodynamics into the Schrödinger form first emerged in the linearized theory of gravitation.<sup>99-101</sup> The state functionals of the linearized theory are defined on embeddings which only slightly differ from the hyperplanes of the flat background. When one uses the Minkowskian coordinates  $X^K = (T, Z^k)$  of the background, one can assume that  $T_{,a}(x)$  and  $Z^k_{,a}(x)$  are small quantities. The intrinsic geometry of a hypersurface  $T = T(x)$  in the Minkowski spacetime depends on  $T(x)$  only through the terms of the second order. On the other hand, the extrinsic curvature  $K_{ab}(x)$  is influenced by a small deformation from the hyperplane already in the terms linear in the deformation:

$$K_{ab}(x) = T_{,ab}(x) . \quad (6.5)$$

It is thus easier to determine  $T(x)$  by looking at the extrinsic curvature of the hypersurface rather than at its intrinsic geometry.<sup>101</sup> A deformation  $Z^k(x)$  of the Cartesian coordinates on the hyperplane, however, affects the intrinsic metric in the first order,

$$g_{ab}(x) = \delta_{ab} + Z_{(a,b)}(x) . \quad (6.6)$$

This enables one to reconstruct  $Z^k(x)$  from the metric.

In linearized gravity, the true degrees of freedom are superimposed on the coordinate-induced effects (6.5) and (6.6). To separate them from the embedding variables, one uses the decomposition of symmetric tensors  $f_{ab}(x)$  into transverse traceless  $f^{\text{TT}}_{ab}$ , transverse  $f^{\text{T}}_{ab}$  and longitudinal  $f^{\text{L}}_{ab}$  parts with respect to the flat background:

$$\begin{aligned} f_{ab} &= f^{\text{TT}}_{ab} + f^{\text{T}}_{ab} + f^{\text{L}}_{ab} , \\ f^{\text{TT}}_{ab,b} &= 0 = f^{\text{TT}}_{ab} , \quad f^{\text{T}}_{ab,b} = 0 , \end{aligned} \quad (6.7)$$

with

$$\begin{aligned}
 f^L_{ab} &= f_{(a,b)} , \quad f_a := \Delta^{-1}(f_{ab,b} - \frac{1}{2} \Delta^{-1} f_{bc,bca}) ; \\
 f^T_{ab} &= \frac{1}{2}(f^T \delta_{ab} - \Delta^{-1} f^T_{,ab}) , \quad f^T := f_{aa} - \Delta^{-1} f_{ab,ab} ; \\
 f^{TT}_{ab} &= f_{ab} - f^T_{ab} - f^L_{ab} .
 \end{aligned} \tag{6.8}$$

(The indices are raised and lowered by the background metric  $\delta_{ab}$ , and  $\Delta^{-1}$  is the inverse Laplacian.)

The decomposition (6.7) - (6.8) can be applied to the linearized canonical variables  $h_{ab} = g_{ab} - \delta_{ab}$  and  $p_{ab}$ . Equations (6.5) and (6.6) which are valid in the Minkowski spacetime can be written as

$$T(x) = - \frac{1}{2} \Delta^{-1} p^T(x) , \quad Z^k(x) = h^k(x) . \tag{6.9}$$

One adopts them as definitions of the embedding variables in linearized gravity. The first equation (6.9) represents a choice of extrinsic time.

The conjugate momenta to the canonical coordinates (6.9) are

$$P_T = \Delta h^T(x) , \quad P_k(x) = - 2 (\Delta p_k(x) + p_{i,ik}(x)) . \tag{6.10}$$

The canonical transformation (1.5) which separates the embedding variables from the true dynamical degrees of freedom takes the form

$$\begin{aligned}
 h_{ab}(x) , p^{ab}(x) &\rightarrow T(x) , Z^k(x) , h^{TT}_{ab}(x) ; \\
 &P_T(x) , P_k(x) , p^{TT\ ab}(x) .
 \end{aligned} \tag{6.11}$$

The meaning of the quantities  $p^T$  and  $h^k$  as parameters specifying the choice of spacetime coordinates was discovered by Arnowitt, Deser, and Misner (ADM).<sup>99,100</sup>

To exclude these surplus variables, ADM fixed the foliation and the reference frame by the coordinate conditions

$$T(x) = t = \text{const} , \quad Z^k(x) = x^k . \tag{6.12}$$

In the many-fingered time formalism, one prefers to keep the variables (6.9) and (6.10) in the theory.

In linearized gravity, the constraints are to be written up to the quadratic

terms in the perturbations of  $g_{ab}$  and  $p^{ab}$ . When expressed in the new canonical variables (6.11), they are to this order linear in the embedding momenta. Their imposition on the states  $\Psi[T,Z,h^{TT}]$  leads then to a functional Schrödinger equation.<sup>101</sup>

Arnowitz, Deser and Misner proposed to use the extrinsic time (6.9) even for large perturbations. Unfortunately, the flat background used in the decomposition (6.7) - (6.8) loses then its geometric and physical significance, and the constraints written in the new variables become unmanageable

▪ **Cylindrical Gravitational Waves.**<sup>35,102</sup> Cylindrical symmetry allows an infinite number of gravitational degrees of freedom and, by permitting  $\infty^1$  many deformations of a hypersurface, it does not fix a single foliation. There are, however, natural spacetime coordinates  $T,R$  (the Einstein-Rosen time and radial coordinates) in which the vacuum Einstein equations for the cylindrical wave assume a simple form. The Einstein-Rosen radial coordinate is simply the function  $R(r)$  characterizing the intrinsic metric (3.1). The Einstein-Rosen time can be reconstructed from the function  $p_\gamma(r)$  characterizing the extrinsic curvature:

$$T(r) = T(\infty) - \int_{\infty}^r dr p_\gamma(r) . \quad (6.13)$$

It is a good example of extrinsic time. One can complete Eq. (6.13) into a canonical transformation

$$\gamma, R, \psi; p_\gamma, p_R, \pi_\psi \rightarrow T, R, \psi; P_T, P_R, \pi_\psi \quad (6.14)$$

by putting

$$\begin{aligned} P_T &= -\gamma' + (\ln(R'^2 - p_\gamma^2))' \quad \text{and} \\ P_R &= p_R + (\ln(R' - p_\gamma)/(R' + p_\gamma))'. \end{aligned} \quad (6.15)$$

The midisuperspace restrictions (3.2) - (3.3) of the super-Hamiltonian and supermomentum constraints are linear in the  $2\infty^1$  momenta  $P_T$  and  $P_R$  and they are thus easily resolved:

$$\begin{aligned} P_T + (R'^2 - T'^2)^{-1} \left[ \frac{1}{2} (R^{-1} \pi_\psi^2 + R \psi'^2) R' - \psi' \pi_\psi T' \right] &= 0 , \\ P_R + (R'^2 - T'^2)^{-1} \left[ -\frac{1}{2} (R^{-1} \pi_\psi^2 + R \psi'^2) T' + \psi' \pi_\psi R' \right] &= 0 \end{aligned} \quad (6.16)$$

The resulting constraints have the same form as those for a massless scalar field

$\psi$  propagating on a flat Minkowskian background which is parametrized by the introduction of arbitrary curved spacetime hypersurfaces  $T(r)$ ,  $R(r)$ . This procedure illustrates how an appropriate choice of the embedding variables can simplify the originally unwieldy constraints.

The canonical quantization based on the constraints (6.16) leads to a  $2\omega^1$ -fingered time Schrödinger equation. For the adopted foliation (6.12), this equation reduces to an ordinary Schrödinger equation for a single massless scalar field in Minkowski space.

▪ **The Mean Extrinsic Curvature as Time.** The simplifications achieved by a judicious choice of extrinsic time in some midisuperspace models and in linearized gravity encourages the search for extrinsic time in full geometrodynamics. An obvious choice seems to be

$$T(x) := \frac{2}{3} |g(x)|^{-1/2} p(x), \quad P(x) := - |g(x)|^{1/2}; \quad (6.17)$$

indeed,  $T(x)$  and  $P(x)$  are a set of  $2\omega^3$  canonically conjugate variables. Dirac proposed that gravity in asymptotically flat spacetimes should be quantized on maximal hypersurfaces  $T(x) = 0$ .<sup>94</sup> The choice (6.17) was elaborated into an effective and elegant scheme of solving the classical constraint equations (1.3) - (1.4) by York and his collaborators.<sup>103-105</sup>

Equations (6.17) can be completed into a canonical transformation by introducing the conformal metric  $\sigma_{ab}$  and its conjugate momentum  $\pi^{ab}$ :

$$\sigma_{ab} := |g|^{-1/4} g_{ab}, \quad \pi^{ab} := |g|^{1/4} (p^{ab} - \frac{1}{3} p g^{ab}). \quad (6.18)$$

Each of these quantities has only  $5\omega^3$  independent components. To simplify the description I shall not attempt to split them further into the  $3\omega^3$  frame variables  $Z^k(x)$  and  $2\omega^3$  gravitational degrees of freedom, in contrast to what is done in York's theory.

In terms of the new variables  $T$  and  $P$ , the super-Hamiltonian constraint (1.2), (1.4) becomes a quasi-linear elliptic equation for  $\phi := P^{1/6}$ :

$$\Delta_\sigma \phi - \frac{1}{8} R[\sigma] \phi + \frac{1}{8} \pi_{ab} \pi^{ab} \phi^{-7} - \frac{3}{64} T^2 \phi^5 = 0. \quad (6.19)$$

Assume that one can find the solution  $\phi(x; T, \sigma, \pi)$  of this equation and hence the corresponding Hamiltonian density  $h = -\phi^6$ . One can then replace the super-

once when it is recontracting. (This was often adduced as a reason why an extrinsic time is to be preferred to an intrinsic time). The global problems of the maximal and  $K = \text{const}$  foliations have been studied in the literature.<sup>108-114</sup> Still, very little is known about the global problem of time in full classical geometrodynamics. These topics definitely deserve more attention.

- **Spectral Analysis Problem.** The solution of the constraints with respect to the embedding momenta may not exist for all values of the dynamical variables and, even if it does, it may be too complicated. One can notice this already on minisuperspace models,<sup>115,96</sup> like the mixmaster universe: the constraint (3.5) contains a square root and, because the potential  $V(\beta_+, \beta_-) - 1$  is negative in the region close to the origin of the  $\beta_+, \beta_-$  plane, the expression under the square root becomes negative for some values of the variables  $\beta_+, \beta_-$ ,  $p_+, p_-$ . In quantum theory, one is called to replace the canonical variables  $\beta_{\pm}$ ,  $p_{\pm}$  by operators and give a meaning to the square root. This can be done in two steps, by first defining the operator under the square root, and then defining the square root itself by *spectral analysis*. There are two difficulties with this procedure. First, if the operator under the square root is not a positive definite operator, spectral analysis yields a Hamiltonian which is not self-adjoint. The Schrödinger equation (2.7) then does not yield unitary evolution. Second, even if the operator under the square root were a positive definite operator, spectral analysis is an involved nonlocal procedure which requires solving an eigenvalue problem for the operator in question. It may be difficult to write such a solution in an explicit form.

The spectral analysis problem becomes even more severe in full geometrodynamics because finding the many-fingered Hamiltonian  $h_A(x; X, \phi, \pi]$  often requires solving nonlinear differential equations and because the definition of the operators involves renormalization problems. To give an example, if one tries to convert the York algorithm into a Schrödinger equation, one faces the problem of defining the operator version of the classical expression (6.20), whose dependence on the dynamical variables  $\sigma_{ab}$  and  $\pi^{ab}$  is known only implicitly, through the statement that the conformal factor  $\phi$  must satisfy the Lichnerowicz-York equation (6.19).

The study of midisuperspace models and covariant field systems like string models strongly indicates that, if there is an internal time which converts the old constraints into the Schrödinger form, such a time is a nonlocal functional of the geometric variables. It is also likely that the many-fingered Hamiltonian  $h_A(x; X, \phi, \pi]$  corresponding to such a choice of time is defined only implicitly, by solving PDE's or by an even more complicated procedure, and that the resulting expression is a rather horrendous functional of the new dynamical variables. This makes the whole program of interpreting quantum gravity through

Hamiltonian constraint (1.4) by an equivalent constraint

$$\Pi(x) := P(x) + h(x; T, \sigma, \pi) = 0 \quad (6.20)$$

whose quantum version is a functional Schrödinger equation<sup>33</sup>

$$i \frac{\delta \Psi[T, \sigma]}{\delta T(x)} = h(x; T, \hat{\sigma}, \hat{\pi}) \Psi[T, \sigma] . \quad (6.21)$$

(The task of separating the frame variables  $Z^k(x)$  from the gravitational degrees of freedom and solving simultaneously the supermomentum constraints for the conjugate momenta is much more involved.<sup>103-105</sup> The equations decouple only if one fixes the foliation:  $T = t = \text{const.}$ )

Having summarized the situations in which the identification of an internal time variable proved useful in quantum canonical gravity, I must turn to the problematic features of such a procedure. The two main problems, the global time problem and the multiple choice problem, representing the opposites of the embarrassment of poverty and the embarrassment of riches, were already identified in Sections 1 and 2. Besides them, there are more specific difficulties which remain to be pointed out.

- **Global Time Problem.** It may happen that the constraint system (1.6) cannot be made globally equivalent to the constraint system (1.3) - (1.4) by any choice of internal time. This must happen if there does not exist a global choice of the time function on the phase space such that each classical trajectory intersects every hypersurface of constant internal time once and only once. (This is the analogue of the Gribov problem<sup>106</sup> for parametrized dynamical systems). These problems were studied for simple model systems - a couple of harmonic (antiharmonic) oscillators in a stationary state (Section 3)<sup>10-12</sup> and in homogeneous cosmologies<sup>13</sup> - by Hájíček. The analysis of the harmonic oscillator systems reveals that many such models do not allow a global time function, and that such models get more and more frequent as the number of the oscillators increases. It is also known that some homogeneous cosmological models - e.g., the Friedmann universe driven by a massive scalar field - do not possess a global time function.<sup>13,107</sup> A systematic analysis of the global time problem for homogeneous cosmological models was not done, but it seems likely that the lack of a global time function is quite generic within this class. Further, even if such a function exists for a given model, it is fairly obvious that some popular choices of the time function are not globally valid. Thus, e.g., the volume time (3.4) is not globally permitted in oscillating cosmological models, because the universe attains a given value of  $\Omega < \Omega_{\text{MAX}}$  at least twice, once when it is expanding, and

an identification of internal time rather remote and unattractive.

- **Multiple Choice Problem.** If there is not a geometrically natural choice of internal time, all choices of time are equally good or equally bad. One faces then the problem of comparing quantum theories which follow from different choices of an internal time variable.

The multiple choice problem can be neatly analyzed on the model (3.6) of a relativistic particle moving in a Riemannian spacetime.<sup>95</sup> Any time function  $T(Q)$  whose level surfaces are spacelike and foliate  $\mathcal{M}$  has the property that each dynamical trajectory of the particle (which is timelike) intersects each leaf of the foliation  $T(Q) = \text{const}$  (which is spacelike) once and only once. There is no global problem of time for a relativistic particle, but there is a multiple choice problem: There are infinitely many permissible time functions  $T(Q)$ . The time functions are constructed from the configuration variables  $Q$ ; they are analogous to different choices of intrinsic time in canonical geometrodynamics.

Construct the reference frame of worldlines orthogonal to the leaves of the foliation  $T(Q) = \text{const}$  and label the worldlines by the comoving coordinates  $q^a$ . The normal separation of the leaves of the  $T(Q) = \text{const}$  foliation is then characterized by the lapse function  $N(Q)$ . Perform the point transformation

$$Q^A, P_A \rightarrow T, P_T; q^a, p_a \quad (6.22)$$

on the phase space of the particle and solve the constraint (3.6) with respect to the momentum  $P_T$ :

$$\begin{aligned} H_T &:= P_T + h_T(T, q, p) = 0, \\ h_T &:= \left( N^2(T, q) (h^{ab}(T, q) p_a p_b + M^2 V(T, q)) \right)^{1/2}, \end{aligned} \quad (6.23)$$

here,  $h_{ab}$  is the intrinsic metric on the leaves of the  $T = \text{const}$  foliation. For future-oriented timelike trajectories, the square-rooted constraint (6.23) is equivalent to the quadratic constraint (3.6). So are any two square-rooted constraints adapted to two different foliations  $T_1$  and  $T_2$ . The dynamical evolutions generated by all such constraints are mutually consistent. This is ensured by the closure of the Poisson brackets

$$\begin{aligned} \{H_{T_1}, H_{T_2}\} &= \text{"C"} H_{T_1} H_{T_2} + \text{"C"} H, \\ \{H_T, H\} &= \text{"C"} H_T + \text{"C"} H. \end{aligned} \quad (6.24)$$

The structure functions "C" are fairly hideous, but they are all regular on the constraint surface. There is no problem of time in classical relativistic mechanics.

One can turn the dynamical variables  $q^a$ ,  $p_a$  into operators and define the Hamiltonians  $h_T$  by spectral analysis. One can then impose the constraints (6.23) as restrictions on the state functions  $\Psi(Q)$ :

$$\hat{H}_{T_1} \Psi = 0 \Rightarrow \Psi \in \mathcal{F}_1, \quad (6.25)$$

$$\hat{H}_{T_2} \Psi = 0 \Rightarrow \Psi \in \mathcal{F}_2. \quad (6.26)$$

One obtains thereby the Schrödinger equations based on the foliations  $T_1(Q) = \text{const}$  and  $T_2(Q) = \text{const}$ . I have denoted the spaces of solutions of these Schrödinger equations by  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , respectively. One can also impose the original constraint (3.6) on the states  $\Psi$  and obtain thereby the Klein-Gordon equation

$$\hat{H}\Psi = 0 \Rightarrow \Psi \in \mathcal{F}_0 \quad (6.27)$$

whose solution space is called  $\mathcal{F}_0$ .

For different choices of the *intrinsic time*  $T(Q)$ , the spaces  $\mathcal{F}_1$ ,  $\mathcal{F}_2$ , and  $\mathcal{F}_0$  can be compared with each other, because each  $\Psi(Q)$  can be considered to be a scalar function on  $\mathcal{M}$ . The multiple choice problem arises because these spaces do not coincide. Technically, one cannot find a factor ordering of the constraints which, based purely on the geometric structures used when writing down each individual constraint, would make the redundant constraint system (6.25) - (6.26) consistent, i.e., which would take the classical closure relations (6.24) into commutator relations in which the operators  $\hat{H}_{T_1}$ ,  $\hat{H}_{T_2}$ , and  $\hat{H}$  on its right-hand side would act on  $\Psi$  before the "C"'s and annihilate it. If the foliations  $T_1$  and  $T_2$  are such that they share the same initial and the same final hypersurface, the initial state  $\Psi_{\text{IN}}$  evolved by the Schrödinger equation (6.25) yields in general a different final state  $\Psi_{\text{FIN}}$  from that obtained by evolving  $\Psi_{\text{IN}}$  by the Schrödinger equation (6.26). The quantum mechanics (6.25) can in principle *be compared* with the quantum mechanics (6.26) and they are in general found to be mutually *incompatible*. When the two choices of the time function are based on an *extrinsic time*, it is even difficult to see how one can *compare* the ensuing quantum mechanical schemes. The multiple choice problem for the intrinsic time functions can be resolved only for a particle which moves in a stationary spacetime with a stationary potential  $V$ ; the solution depends on the existence of a privileged background structure, namely, a timelike Killing vector field.

The multiple choice problem in canonical geometrodynamics is at least as serious as the multiple choice problem explained here for the relativistic particle.



• **Spacetime Problem.** I pointed out that time in general relativity is a function variable rather than the single parameter of Newtonian mechanics. In classical geometrodynamics, one is asked to reconstruct the time coordinate  $T$  of an event  $X$  in an Einstein spacetime  $(\mathcal{M}, \gamma)$  from the canonical data  $g_{ab}(x), p^{ab}(x)$  on an embedding  $X(x)$  passing through  $X$ . The time should emerge from the canonical formalism as a spacetime scalar which does not depend on the embedding from which it was reconstructed. Thus, if two embeddings intersect at the same event  $X$  and if the canonical data on these two embeddings are related by Einstein's equations, the internal time  $T(x; g, p)$  obtained from such two sets of the canonical data should be the same.<sup>95</sup> This imposes the requirement

$$\{T(x), \int_{\Sigma} d^3x' H(x')N(x')\} \approx 0 \quad \text{for } \forall N(x'): N(x) = 0 \quad (6.28)$$

on the possible candidates  $T(x; g, p)$  for an internal time function.<sup>116,117</sup> The same limitation, of course, should also be imposed on the internal spatial coordinates  $Z^k(x; g, p)$ . To find internal embedding variables which satisfy Eq. (6.28) constitutes the *spacetime problem*.

The requirement (6.28) is indeed quite strong. It is fairly obvious that it excludes any intrinsic time  $T(x; g)$  which, like  $R(x; g)$ , is a local functional of the three-geometry. It also excludes some obvious choices of extrinsic time. Thus, e.g., the time function (6.17) is blatantly not a spacetime scalar: When one bends a hypersurface around a given event in a vacuum Einstein spacetime, the extrinsic curvature  $\frac{1}{2}|g|^{-1/2}p$  changes even if the event remains the same. The time function (6.17) helps one to define a unique foliation  $T(x) = t = \text{const}$  of a vacuum Einstein spacetime by hypersurfaces of constant mean extrinsic curvature, but the value of  $T(x)$  on a hypersurface which is transverse to that foliation gives the mean extrinsic curvature on that hypersurface, not the mean extrinsic curvature on the leaf of the foliation passing through the same event. The canonical coordinate (6.17) thus cannot be turned into a coordinate in spacetime. For this reason, the time variable (6.17) is unsuitable as a many-fingered time variable in the functional Schrödinger equation (2.7).

It should in principle be possible to take the data  $g_{ab}, p^{ab}$  on an embedding and construct from them a time function  $T_o(x; g, p)$  whose value at  $x$  would equal (4/3 of) the mean extrinsic curvature of the hypersurface of constant mean extrinsic curvature passing through the same event. Such a time function would be a spacetime scalar. However, one can expect  $T_o(x; g, p)$  to be a hideously nonlocal functional of the hypersurface data  $g_{ab}, p^{ab}$ . Essentially, the construction of  $T_o(x; g, p)$  calls for building the whole spacetime by the evolution of the initial data  $g_{ab}, p^{ab}$ , finding in this spacetime the leaves of constant mean extrinsic

curvature, and assigning this constant mean curvature to the points at which the leaves intersect the initial hypersurface.

There are, of course, local functionals  $F(x; g, p]$  of the canonical data which are spacetime scalars. As an example, one can take the square  $R_{\alpha\beta\gamma\delta}(X)R^{\alpha\beta\gamma\delta}(X)$  of the Riemann curvature tensor of a vacuum Einstein spacetime, and reexpress it in terms of the canonical data on a spacelike hypersurface. However, such an  $F(x)$  can be a candidate for the time function  $T(x)$  only if  $\{F(x), F(x')\} = 0$  and if the hypersurfaces  $F(x) = \text{const}$  are spacelike. Even then, to use  $F(x)$  as a time function one must split it from the rest of the canonical variables by a canonical transformation, which is by no means a trivial task. So far, I do not know a single concrete example of a decomposition of the fundamental canonical variables based on a scalar time function  $T(x)$ .

## 7. Matter Clocks and Reference Fluids

The intrinsic geometry and extrinsic curvature of a spacelike hypersurface enter into the constraints in a complicated way. Nothing in the structure of the constraints tells us how to distinguish the true dynamical degrees of freedom from the quantities which determine the hypersurface. This makes the task of constructing an internal time from the geometric variables so difficult and ambiguous.

The main role of embedding variables is to provide an orientation in spacetime, i.e., to identify spacetime events. The founding fathers of general relativity invented a conceptual device which does exactly that: the notion of a *reference fluid*. The particles of the fluid identify space points, and clocks carried by them identify instants of time. These fix the reference frame and the time foliation. In that frame and on that foliation, the metric itself, rather than the geometry, becomes measurable.

The concept of reference fluid goes back to Einstein,<sup>118</sup> who coined for it the charming term "molusc", and to Hilbert who, in the second of his famous communications on foundations of physics,<sup>119</sup> formalized the idea that the coordinate system should be realized by a realistic fluid carrying clocks which keep a causal time: He imposed a set of inequalities ensuring that the worldlines of the reference frame be timelike and the leaves of the time foliation be spacelike.

The reference fluid is traditionally considered as a tenuous material system whose back reaction on the geometry can be neglected. There is just enough matter to tell us where we are but not enough of it to disturb the geometry. Instead of deriving the motion of the fluid from its action, one encodes it in the coordinate conditions. These are statements about the metric which hold in the

coordinate system of the fluid and are violated in any other system. Unfortunately, such a standpoint makes it difficult to view the reference fluid as a physical object which, in quantum gravity, could assume the role of an apparatus for identifying spacetime events.

To turn the reference fluid into a physical system, one can follow one of two alternative routes. The first one is to picture the fluid as a realistic material medium and devise a Lagrangian which describes its properties. By adding this Lagrangian to the Hilbert Lagrangian, one couples the fluid to gravity. The second route imposes the coordinate conditions *before* variation by adjoining them to the action with Lagrange multipliers. The additional terms in the action are then parametrized and interpreted as a matter source.

▪ **Phenomenological Fluids in Quantum Gravity.** The first route of using phenomenological fluids in quantum gravity was pioneered by DeWitt. In the covariant approach, DeWitt coupled the gravitational fluid to an elastic medium carrying mechanical clocks.<sup>120</sup> From these objects he constructed idealized apparatuses which, in the spirit of the Bohr-Rosenfeld analysis of measurability in quantum electrodynamics,<sup>121</sup> were able to detect appropriate projections of the quantized Riemann curvature tensor. A little later, he used the same device for interpreting the canonical minisuperspace quantization of the Friedmann universe:<sup>32</sup> he introduced a cloud of clocks into the model and studied their correlations with the radius of the universe. DeWitt's approach was recently rediscovered by Rovelli.<sup>122</sup> Rovelli used a square-root Hamiltonian for the clocks. He argued that when the clock's momentum is large in comparison with its mass, the Dirac constraint quantization approximately leads to a Schrödinger equation.

▪ **Reference Fluids Associated With Coordinate Conditions.** The association of a material medium with coordinate conditions stems from the problem of representing spacetime diffeomorphisms in canonical gravity.<sup>123-126</sup> Isham and Kuchar<sup>127-129</sup> approached this problem by breaking the invariance of general relativity by Gaussian coordinate conditions and restoring it again by parametrization. This procedure leads to the modification of the constraints by terms which are linear in the momenta conjugate to the Gaussian coordinates. Hartle<sup>130</sup> discussed the Schrödinger equation obtained by imposing the new constraints as restrictions on the physical states. In the end, he dismissed the new terms as devoid of physical reality. Halliwell and Hartle<sup>131</sup> related the new form of the constraints to the sum-over-histories approach to quantum gravity. Kuchař and Torre<sup>28</sup> showed that the modified constraints follow from an action principle when one enforces the Gaussian coordinate conditions with Lagrange multipliers, and that the new terms can be identified with energy-momentum densities of the

corresponding reference fluid. This method is applicable to any coordinate conditions. The harmonic coordinate conditions were treated by Kuchař and Torre<sup>132</sup> and Kuchař and Stone.<sup>133</sup> The  $K = \text{const}$  slicing condition was related to a reference fluid by Kuchař.<sup>134</sup> The same approach can be used for introducing the conformal, harmonic, and light-cone gauges in the canonical theory of a bosonic string<sup>81</sup> and in the canonical treatment of (induced) two-dimensional gravity.<sup>135</sup> I shall describe the Schrödinger interpretation of quantum gravity based on the matter clock associated with a Gaussian reference fluid.

■ **Gaussian Reference Fluid.** In the normal Gaussian coordinates  $X^K = (T, Z^k)$  four components of the spacetime metric  $\gamma^{KL}$  are fixed by the conditions

$$\gamma^{00} + 1 = 0, \quad \gamma^{0k} = 0. \quad (7.1)$$

These follow from the action

$$S^F[\gamma_{KL}, M, M_k] = \int_{\mathcal{M}} d^4X |\det(\gamma_{KL})|^{1/2} \left( -\frac{1}{2}M(\gamma^{00} + 1) + M_k \gamma^{0k} \right) \quad (7.2)$$

by varying the Lagrange multipliers  $M_K = (M, M_k)$ . By expressing the privileged (Gaussian) coordinates  $X^K$  as functions of arbitrary (label) coordinates  $x^\alpha$ ,  $X^K = X^K(x^\alpha)$ , one arrives at the parametrized action

$$S^F[\gamma_{\alpha\beta}, M_K, X^K] = \int_{\mathcal{M}} d^4x |\det(\gamma_{x\lambda})|^{1/2} \left( -\frac{1}{2}M(\gamma^{\alpha\beta} T_{,\alpha} T_{,\beta} + 1) + M_k \gamma^{\alpha\beta} T_{,\alpha} Z^k_{,\beta} \right). \quad (7.3)$$

This is added to the Hilbert action  $S^G[\gamma_{\alpha\beta}]$  of vacuum gravity. The total action

$$S[\gamma_{\alpha\beta}, M_K, X^K] = S^G[\gamma_{\alpha\beta}] + S^F[\gamma_{\alpha\beta}, M_K, X^K] \quad (7.4)$$

describes the Gaussian reference fluid coupled to gravity.

The variation of the total action (7.4) with respect to  $\gamma_{\alpha\beta}$  leads to the Einstein law of gravitation

$$G^{\alpha\beta} = \frac{1}{2} T^{\alpha\beta} := |\gamma|^{-1/2} \delta S^F / \delta \gamma_{\alpha\beta}, \quad (7.5)$$

in which there appears a source,  $T^{\alpha\beta}$ , the energy-momentum tensor of the reference fluid. The variation of the multipliers  $M_K$  yields the parametrized Gaussian coordinate conditions. The variation of the function variables  $X^K(x^\alpha)$  gives the Euler hydrodynamic equations of the reference fluid. The variables

$X^K(x^\alpha)$  thus play the role of the velocity potentials of the fluid.<sup>136,137</sup> Because the action (7.4) is invariant under arbitrary transformations of  $x^\alpha$ , the Euler hydrodynamic equations follow from the Einstein law and the Gaussian coordinate conditions. They amount to the conservation law  $\nabla_\beta T^{\alpha\beta} = 0$  of energy and momentum of the reference fluid.

By introducing the mutually orthogonal fields

$$U^\alpha := -\gamma^{\alpha\beta} T_{,\beta} \quad \text{and} \quad M_\alpha := M_K Z^k_{,\alpha} , \quad (7.6)$$

the energy-momentum tensor (7.5) takes the form

$$T^{\alpha\beta} = M U^\alpha U^\beta + M^{(\alpha} U^{\beta)} \quad (7.7)$$

of the Eckart energy-momentum tensor of a heat-conducting fluid.<sup>138,139</sup> The four velocity  $U^\alpha$  of the fluid is the four-velocity of the Gaussian reference frame, the multiplier  $M$  has the meaning of the *proper mass density*, and the spacelike vector  $M^\alpha$  is the *heat flow*. If the heat flow vanishes, the reference fluid turns into an incoherent dust.

▪ **Energy Conditions.** The use of Gaussian conditions as auxiliary conditions in the Hilbert action principle is a purely formal device. It is quite surprising that the energy-momentum tensor which emerges from this procedure seems to describe a simple physical system, namely, heat conducting dust. If we had such a material at our disposal and scattered it throughout space, we could identify the Gaussian coordinates with its physical state variables. Figuratively speaking, we could anchor the Gaussian coordinates to the physical worldlines of the dust. However, unlike an abstract Gaussian system, the dust can be real only if its energy-momentum tensor satisfies appropriate energy conditions.<sup>140</sup>

For the energy-momentum tensor (7.7), the weak, the strong, and the dominant energy conditions are all equivalent, and they are satisfied if the multipliers  $M_K$  satisfy the inequality

$$M - 2|M| \geq 0 , \quad |M|^2 := \gamma^{\alpha\beta} M_\alpha M_\beta . \quad (7.8)$$

Unfortunately, even if Eq. (7.8) holds at one instant, it can be violated later by the dynamical evolution. Only if the heat flow vanishes, are the energy conditions preserved in time. This foreshadows the problems encountered in interpreting the Schrödinger equation obtained by quantizing geometry whose source is the Gaussian reference fluid.

▪ **Canonical Description of the Gaussian Reference Fluid.** As in all

theories with nonderivative gravitational coupling, the canonical description of gravity interacting with the reference fluid is obtained by adding to the gravitational super-Hamiltonian (1.2) (which we now call  $H^G(x)$ ) the energy density  $H^F(x)$  of the fluid, and to the gravitational supermomentum (1.1) (which we now call  $H^G_a(x)$ ) the momentum density  $H^F_a(x)$  of the fluid, and by imposing the constraints

$$H := H^G + H^F = 0, \quad H_a := H^G_a + H^F_a = 0 \quad (7.9)$$

The standard ADM analysis yields

$$H^F_a(x) = X^K_{,a}(x)P_K(x) \quad \text{and} \quad H^F(x) = n^K(x; X] P_K(x), \quad (7.10)$$

with

$$n^K = \left( (1 + g^{ab}T_{,a}T_{,b})^{1/2}, (1 + g^{ab}T_{,a}T_{,b})^{-1/2}g^{cd}T_{,c}Z^k_{,d} \right). \quad (7.11)$$

The densities (7.10) are functionals of the embedding variables  $X^K(x)$ , the embedding momenta  $P_K(x)$ , and the intrinsic metric  $g_{ab}(x)$ . The Lagrange multipliers were eliminated in the process.

The spectacular feature of Eq. (7.10) is that not only the momentum density  $H^F_a(x)$ , but also the energy density  $H^F(x)$  is a linear homogeneous function of the embedding momenta  $P_K(x)$ . It is thus straightforward to solve the constraints (7.9) with respect to  $P_K(x)$ . The only thing one needs to do is find the cobasis  $(-n_K, X^K_a)$  to the basis  $(n^K, X^K_{,a})$ . Geometrically, the coefficients (7.11) are the components of the unit normal to the embedding  $X^K(x)$  in the Gaussian coordinate system  $X^K$ , and  $X^K_{,a}$  are the tangent vectors to the embedding. The cobasis is formed by the dual covectors.

This brings the constraints into the form (1.6) on the extended phase space  $X^K, P_K, g_{ab}, p^{ab}$ , with

$$\begin{aligned} h_K(x; X, g, p] &= -n_K(x; X, g] H^G(x; g, p] \\ &+ X^K_a(x; X, g] H^G_a(x; g, p]. \end{aligned} \quad (7.12)$$

▪ **Probabilistic Interpretation.** By imposing the constraints as operator restrictions on the state functional  $\Psi[X, g]$ , I get the many-fingered time Schrödinger equation. This suggests that the expression

$$|\Psi[T(x), Z^k(x), g_{ab}(x)]|^2 \quad (7.13)$$

constructed from a solution of this equation can be interpreted as the probability density that, on the embedding  $T(x)$ ,  $Z^k(x)$ , the metric  $g_{ab}(x)$  which is measured in the system of coordinates  $x^a$  connected to the Gaussian frame coordinates  $Z^k$  by the transformation  $Z^k = Z^k(x^a)$ , be found in the cell  $Dg_{ab}$  centered about  $g_{ab}(x)$ .

The advantages of the Gaussian matter time over an internal geometric time are rather obvious:

- Because the energy-momentum densities (7.10) are linear in the embedding momenta  $P_K$ , the many-fingered-time Hamiltonian (7.12) is a local functional of the variables  $X$ ,  $g$ , and  $p$  which is quadratic in  $p^{ab}(x)$ . There is no spectral analysis problem.

- Because the variables  $X^K(x^a) = (T(x^a), Z^k(x^a))$  in the action (7.3) are spacetime scalars, the spacetime problem is automatically solved. Indeed, the embedding variables  $X^K(x)$  satisfy Eq. (6.28).

- The introduction of a material reference frame and time foliation enables one to measure the metric rather than the mere (conformal) geometry.

An internal time based on the Gaussian reference fluid and the interpretation of quantum gravity which it offers come rather close to implementing the internal time program. However, there are two major objections against accepting such an interpretation of quantum gravity:

- **The Problem With Energy Conditions.**<sup>141</sup> The clock variable  $T(x)$  and the frame variables  $Z^k(x)$  are physically realizable only when the reference fluid satisfies the appropriate energy conditions. When the energy conditions are violated, the variables  $T(x)$ ,  $Z^k(x)$  cannot be identified by observing the physical properties of a real system and the interpretation of the expression (7.13) loses a sound epistemological foundation. It is thus necessary to consider the status of the energy conditions (7.8) in quantum gravity.

In the canonical formalism, the weak energy condition simply requires that  $H^F(x) \geq 0$ . The super-Hamiltonian constraint (7.9) and the use of the step function  $\theta$  enables one to express the energy condition as a set of constraints

$$\theta(H^G(x)) = 0 \quad \forall x \in \Sigma \tag{7.14}$$

on the geometric variables. When these constraints are adjoined to the super-Hamiltonian and supermomentum constraints, the total system ceases to be first class:

$$\{\theta(H^G(x)), H(x')\} \neq 0 \text{ and } \{\theta(H^G(x)), \theta(H^G(x'))\} \neq 0. \quad (7.15)$$

The first inequality (7.15) tells us that the energy conditions are not preserved in time. The Poisson brackets (7.15) weakly vanish only if one imposes an additional constraint

$$P_k(x) = 0, \quad (7.16)$$

i.e., if one switches off the heat conduction and turns the Gaussian fluid into an incoherent dust. The constraint (7.16) tells us that the frame coordinates  $Z^k(x)$  are no longer measurable.

When one tries to impose the energy conditions (7.14) as operator restrictions on the states,

$$\theta(\hat{H}^G(x)) \Psi[X, g] = 0, \quad (7.17)$$

they become incompatible with the many-fingered time Schrödinger equation due to the commutator counterparts of Eq. (7.15). The consistency is restored for an incoherent dust, but the operator version of Eq. (7.16) makes the states independent of  $Z^k(x)$ . The interpretation of

$$|\Psi(x; T, g)|^2 \quad (7.18)$$

as the probability density for the geometry  $g(x)$  which is represented on the hypersurface  $T(x)$  by the metric  $g_{ab}(x)$  to be found in the cell  $Dg$  centered about  $g(x)$  is hampered by the fact that the geometry operator  $g(x) \times$  defined as a multiplication operator does not commute with the energy conditions. This does not mean that the probability density for the three-geometry  $g(x)$  does not exist, but that its identification requires a much more complicated and implicit procedure, analogous to the construction of the Newton-Wigner position operators for a relativistic particle.

It can be argued that the energy conditions (7.17) are unnecessarily strong, because they are imposed on energy *densities*, and these can become negative in a quantum field theory. One should keep in mind, however, that geometrodynamics is fundamentally different from an ordinary field theory: there, the classical energy density is positive-definite function of the field variables, while the classical expression  $H^G(x)$  entering into Eq. (7.14) is indefinite.

• **Reference Fluids are Phenomenological Systems.** They can provide an interpretation of quantum gravity in "laboratory conditions," but one can



scarcely expect to find such systems, say, in the early Universe.

▪ **Fundamental Fields as Matter Clocks.** One can counter both of these objections by saying that matter clocks are to be constructed from fundamental matter fields rather than from phenomenological fluids. The energy density of fundamental fields is supposed to satisfy at least the weak energy condition at the classical level. Such fields will always be present, even when phenomenological systems are destroyed by tidal forces. The idea of constructing a matter clock from a massless scalar field was tried in minisuperspace models.<sup>96,13</sup> (Incidentally, the massless scalar field is the reference fluid associated with the harmonic coordinate conditions.<sup>132</sup>)

• **Shortcomings of Fundamental Matter Clocks.** However, when one starts exploring the fundamental matter fields as possible clocks, the advantages provided by a simple phenomenological system, like the Gaussian fluid, start quickly disappearing. The positivity of the classical energy density precludes it from being linear in the momentum conjugate to the field. When solving the super-Hamiltonian constraint with respect to such a momentum, the spectral analysis problem reappears in the quantum theory. Because the solution depends on  $H^G(x)$ , which is not positive-definite, the range of the geometric variables must usually be restricted for the solution to be real. If the matter clock is constructed from any other field than a spacetime scalar, one faces again the spacetime problem. By and large, identifying the embedding variables from among the fundamental matter fields is as complicated as looking for a geometric clock.

## 8. Unspecified Cosmological Constant and Cosmological Time

▪ **Unimodular Gravity.** In one special case, the reference fluid associated with a coordinate condition allows a geometric interpretation. This is the unimodular coordinate condition

$$|\det(\gamma_{KL})|^{1/2} = 1 \quad (8.1)$$

fixing the spacetime volume element. The unimodular coordinates (8.1) were invoked by Einstein in the same paper in which he laid the foundations of the general theory of relativity.<sup>142</sup> By imposing the unimodular condition before rather than after the variation, one obtains the law of gravitation with an unspecified cosmological constant.<sup>143</sup> The reference fluid reduces in this case to the cosmological term. The law of gravitation with an unspecified cosmological constant was introduced by Einstein<sup>144</sup> and recently rediscovered, revived and reviewed by a number of authors.<sup>145-148</sup>

Unimodular gravity was cast into a canonical form by Henneaux and Teitelboim<sup>149</sup> and by Unruh.<sup>143,150</sup> The super-Hamiltonian constraint takes the familiar form

$$\lambda + |g|^{-1/2} H^G(x) = 0, \quad (8.2)$$

except that the cosmological constant  $\lambda$  appears in the formalism as a canonical momentum conjugate to a coordinate  $\tau$ , the "cosmological time." Henneaux and Teitelboim have shown that the increment of the cosmological time equals the four-volume enclosed between the initial and final hypersurfaces.<sup>149</sup> (In a related development, Sorkin used the four-volume time in a path-integral approach to quantum gravity.<sup>151-152</sup>) The action principle of general relativity with the cosmological term can be regarded as analogous to the Jacobi principle of classical mechanics.<sup>153</sup>

▪ **Cosmological Time and the Probabilistic Interpretation.** The appearance of the cosmological time in unimodular gravity raised hopes that this theory can solve the problem of time in quantum gravity. Because the constraint (8.2) is linear in the cosmological constant, its imposition on the state functional leads to an ordinary Schrödinger equation

$$i\partial_\tau \Psi(\tau; g] = (|g|^{-1/2} H^G(x; g, p))^\wedge \Psi(\tau; g] \quad (8.3)$$

Unruh and Wald<sup>76</sup> suggested that any reasonable quantum theory should contain a parameter (which they called Heraclitian time) whose role is to set the conditions for measuring quantum variables and to provide the temporal ordering of such measurements. They surmised that the cosmological time of unimodular gravity is such a parameter. The Schrödinger equation (8.3) implies that the ordinary quadratic norm is conserved in  $\tau$ . It is thus tempting to interpret  $\Psi(\tau; g]$  as the probability amplitude for the outcomes of measurements carried out on geometric variables at a given instant of  $\tau$ .

• **Cosmological Time is not a Functional Time.**<sup>154</sup> The problem with this suggestion is that the cosmological time is not in any obvious way related to the standard concept of time in the theory of relativity. The basic canonical variables  $g_{ab}(x)$  and  $p^{ab}(x)$ , which unimodular gravity shares with geometrodynamics, are always supposed to be measured *on a single spacelike hypersurface* rather than *at a single value of the cosmological time*. To specify a hypersurface, one needs to use embedding variables, i.e., functions of three coordinates, rather than a single real parameter of Newtonian mechanics. It thus remains obscure in what sense the cosmological time "sets the conditions of

quantum measurements."

To clarify this issue, one needs only to adjoin the unimodular condition to the action with a Lagrange multiplier and follow the algorithm developed in the last section. Through the parametrization process, the unimodular coordinates  $X^K = (T, Z^k)$  appear in the action as embedding variables. However, the resulting constraint system is radically different from the single set of  $(3+1)\omega^3$  constraints (7.9) of the Gaussian gravity. It splits into  $3\omega^3 + (\omega^3 - 1)$  constraints on the embedding variables,

$$P_k(x) = 0 \text{ and } (P_T(Z))_{,k} = 0 \quad (8.4)$$

the same number of constraints on the geometric variables,

$$H^G_a(x) = 0 , \quad (8.5)$$

$$(|g(x)|^{-1/2} H^G(x))_{,a} = 0 , \quad (8.6)$$

and one single constraint

$$\begin{aligned} & \int_{\Sigma} d^3Z P_T(Z) / \int_{\Sigma} d^3Z =: \lambda[X,P] \\ & = \lambda^G[g,p] := - \int_{\Sigma} d^3x H^G(x) / \int_{\Sigma} d^3x |g(x)|^{1/2} \end{aligned} \quad (8.7)$$

coupling the geometric variables to the embedding variables.

When the embedding constraints (8.4) are imposed as operator restrictions on the states  $\Psi[X,g]$ , they ensure that  $\Psi$  depends on the unimodular embedding variables  $X^K(x)$  only through a single quantity

$$\tau := \int_{\Sigma} d^3Z T(Z) . \quad (8.8)$$

The parametrized unimodular condition tells us that this is the four-volume enclosed between a fiducial embedding  $T(Z) = 0$  and the embedding  $T(Z)$ . The cosmological time  $\tau$  thus labels the equivalence classes  $\mathcal{E}_\tau$  of embeddings separated from the fiducial embedding by a given four-volume. Each equivalence class contains  $4\omega^3 - 1$  embeddings, i.e.,  $\omega^3 - 1$  hypersurfaces, separated from each other by zero four-volume. The constraints (8.4) are to be interpreted as a statement that the individual embedding within each equivalence class are physically irrelevant.

The supermomentum constraint (8.5) ensures that the state  $\Psi$  depends only on the three-geometry  $g : \Psi = \Psi(\tau; g)$ . The constraint (8.7), which couples the

embedding variables to the geometric variables, yields the Schrödinger equation

$$i\partial_\tau \Psi(\tau; g] = \int_\Sigma d^3x \hat{H}^G(x) / \int_\Sigma d^3x |\hat{g}(x)|^{1/2} \Psi(\tau; g] . \quad (8.9)$$

If this were the only equation on the state, one could interpret  $\Psi$  as the probability amplitude that the geometry  $g(x)$  has a given distribution at the time  $\tau$ . However,  $\Psi$  must also satisfy the additional  $\infty^3 - 1$  geometric constraints (8.6):

$$(|g(x)|^{-1/2} H^G(x))^\wedge_a \Psi(\tau; g] = 0 . \quad (8.10)$$

The three-geometry operator does not commute with these constraints and the interpretation of  $\Psi$  as the probability amplitude becomes untenable. The  $\infty^3 - 1$  geometric constraints (8.10) are actually a Wheeler-DeWitt equation which describes the evolution of states between embeddings of a single equivalence class  $\mathcal{E}_\tau$ . The cosmological time does not set the conditions of the measurement uniquely because it does not tell us on which one of the infinitely many hypersurfaces of the equivalence class the geometric variables are to be measured. The hypersurfaces with different values of  $\tau$  are allowed to intersect and the cosmological time thus does not even provide the causal ordering required of the Heraclitian time. The fundamental reason why unimodular gravity does not and cannot solve the problem of time is simple: time in relativity is a collection of all spacelike hypersurfaces, and no single parameter, such as the cosmological time  $\tau$ , is able to label uniquely so many instants.

## 9. Klein-Gordon Interpretation

The problem of identifying an internal time and handling the ensuing Schrödinger equation is beset with so many difficulties that one may start wondering if the net result is worth the trouble. I shall turn now to those interpretations of quantum gravity which give up the attempts to separate time from the dynamical degrees of freedom and rely instead on the Wheeler-DeWitt equation (2.2), (2.5).

The Wheeler-DeWitt equation is a second-order variational equation for the state functional  $\Psi[g]$ . One can view it as an infinitely-dimensional analogue of the Klein-Gordon equation for the relativistic particle. To illustrate the problems which arise when one tries to interpret the Wheeler-DeWitt equation, it is best to review the probabilistic interpretation of the Klein-Gordon particle.

▪ **The Hilbert Space Problem for a Relativistic Particle.** The super-Hamiltonian of a relativistic particle is given by the expression (3.6). In the Dirac constraint quantization,  $H$  is turned into the Klein-Gordon operator

$$\hat{H} = \frac{1}{2M} \left( - G^{AB}(Q) \nabla_A \nabla_B + M^2 V(Q) \right) \quad (9.1)$$

and imposed as a restriction

$$\hat{H} \Psi(Q) = 0 \quad (9.2)$$

on the physical states. The main task is to endow the space of solutions  $\mathcal{S}_0$  of Eq. (9.2) with an inner product and complete it into a Hilbert space. This constitutes the Hilbert space problem.

The construction of the inner product starts with the current

$$J_{12}^A := \frac{1}{2i} G^{AB} \left( \Psi_1^* \overset{\leftrightarrow}{\partial}_B \Psi_2 \right). \quad (9.3)$$

If  $\Psi_1$  and  $\Psi_2$  lie in  $\mathcal{S}_0$ , the current (9.3) satisfies the continuity equation

$$\nabla_A J_{12}^A = |G|^{-1/2} \partial_A (|G|^{1/2} J_{12}^A) = 0 \quad (9.4)$$

which implies that the integral

$$\langle \Psi_1 | \Psi_2 \rangle := \int_{\Sigma} d\Sigma_A J_{12}^A, \quad (9.5)$$

$$d\Sigma_A := \epsilon_{AA_1 A_2 A_3} d_{(1)}Q^{A_1} d_{(2)}Q^{A_2} d_{(3)}Q^{A_3}$$

does not depend on the choice of the hypersurface  $\Sigma \rightarrow \mathcal{M}$ . Unfortunately, the product (9.5) is not positive definite on  $\mathcal{S}_0$  and hence is unsuitable to serve as a probability.

It is generally acknowledged that the Hilbert space problem for a single relativistic particle has no solution unless the spacetime background  $G^{AB}(Q)$  and the potential  $V(Q)$  are stationary.<sup>155-158</sup> In that case, there exists a timelike future-pointing Killing vector field  $U^A(Q)$ ,

$$(L_U G)_{AB} = - \nabla_{(A} U_{B)} = 0, \quad (9.6)$$

which preserves  $V(Q)$ :

$$L_U V = U^A \partial_A V = 0. \quad (9.7)$$

These conditions imply that the energy

$$E := -U^A(Q) P_A \quad (9.8)$$

of the particle in the stationary reference frame is a constant of the motion,

$$\{E, H\} = 0 . \quad (9.9)$$

Relativistic particles move along future-oriented timelike worldlines. Along such worldlines, the energy (9.8) is always positive,  $E > 0$ .

On quantization, the energy (9.8) is turned into the operator

$$\hat{E} := i(U^A \partial_A + \frac{1}{2} \operatorname{div}_G U) = iU^A \partial_A , \quad (9.10)$$

which, by virtue of Eqs. (9.6) and (9.7), commutes with the Klein-Gordon constraint operator:

$$[\hat{E}, \hat{H}] = 0 . \quad (9.11)$$

This means that the eigenvalue equation for  $E$ ,

$$\hat{E} \Psi = E \Psi , \quad (9.12)$$

and the Klein-Gordon constraint (9.2) have common solutions. One postulates that the quantum particle, like the classical one, must have positive energy. This amounts to saying that the physical space is not the whole space  $\mathcal{S}_0$  of solutions of the Klein-Gordon constraint, but only a linear subspace of  $\mathcal{S}_0$  spanned by the positive-energy solutions. Thus, if  $\mathcal{S}^+$  is spanned by the positive-energy eigenfunctions of Eq. (9.12), and  $\mathcal{S}_0$  is the solution space of Eq. (9.2), the physical space  $\mathcal{S}_0^+$  is the intersection  $\mathcal{S}_0^+ = \mathcal{S}_0 \cap \mathcal{S}^+$ . One can prove that on  $\mathcal{S}_0^+$  the inner product (9.5) is positive definite. One can complete  $\mathcal{S}_0^+$  in the norm given by Eq. (9.5). The statistical interpretation of the theory is then based on this inner product. One can show that this construction of the Hilbert space amounts to replacing the Klein-Gordon equation by a Schrödinger equation.<sup>95</sup>

If there is no Killing vector, there is no energy operator which commutes with  $\hat{H}$ , and the construction of the Hilbert space fails. Physically, one cannot maintain a one-particle interpretation of the theory because the dynamical background starts producing particles. Even if there is a Killing vector, one can prove the positivity of the inner product only if this vector is timelike and the potential  $V$  in the super-Hamiltonian (9.1) is non-negative. This sets the limits within which one knows how to resolve the Hilbert space problem for a relativistic particle.

▪ **The Hilbert Space Problem in Geometrodynamics.** Proceeding by analogy, one can ask how much of this algorithm is applicable to quantum geometrodynamics. The analogue of the Klein-Gordon product (9.5) was introduced by DeWitt.<sup>32</sup> Let  $\Psi_1[g]$  and  $\Psi_2[g]$  be two solutions of the Wheeler-DeWitt equation and

$$d\Sigma^{ab}(x) := \epsilon^{ab a_1 b_1 \dots a_s b_s} d_{(1)}g_{a_1 b_1}(x) \dots d_{(s)}g_{a_s b_s}(x) \quad (9.13)$$

the directed surface element of a hypersurface in superspace. Then

$$\langle \Psi_1 | \Psi_2 \rangle := \frac{1}{2i} \int_{x \in \Sigma} d\Sigma^{ab} G_{abcd}(x; g) (\Psi_1^*[g] \frac{\delta}{\delta g_{cd}(x)} \Psi_2[g]) \quad (9.14)$$

At least formally, the inner product (9.14) does not depend on the choice of the hypersurface in superspace. As in particle dynamics, this product is not positive in the space  $\mathcal{S}_0$  of solutions to the Wheeler-DeWitt equation. One must ask: Does the super-Hamiltonian (1.2) satisfy the conditions which would allow one to restrict  $\mathcal{S}_0$  to a linear subspace  $\mathcal{S}_0^+$  on which the product (9.14) is positive definite? Unfortunately, there are several reasons why the algorithm developed for the relativistic particle fails:

- **Super-Hamiltonian is not Stationary.**<sup>33,159,160</sup> To decide whether the supermetric  $G_{abcd}(x; g)$  and the potential  $-|g|^{1/2}R(x; g)$  admit an "isometry," one should look for a dynamical variable

$$E := - \int_{\Sigma} d^3x u_{ab}(x; g) p^{ab}(x) \quad (9.15)$$

that is linear in the momentum  $p^{ab}(x)$  and has at least a weakly vanishing Poisson bracket with the super-Hamiltonian (1.2):

$$\{E, H(x)\} \approx 0. \quad (9.16)$$

Indeed,  $E$  should also have a vanishing Poisson bracket with the supermomentum (1.1), but this goal is easily met by requiring that the coefficient  $u_{ab}(x; g)$  in Eq. (9.15) be a spatial tensor field.

It is quite difficult to prove whether such a variable exists or not (Kuchar<sup>159</sup>), the trouble being that the "Killing vector field"  $u_{ab}(x; g)$  in the space of Riemannian metrics may in principle be a highly nonlocal functional of  $g$ . The proof relies on the fact that the super-Hamiltonian  $H(x)$  and the supermomentum  $H_a(x)$ , through which the weak equality in Eq. (9.16) is enforced, have quite

definite locality and polynomial structures. Thus  $H_a(x)$  is linear in the momentum and contains at most the first derivatives of the canonical variables  $g_{ab}(x)$ ,  $p^{ab}(x)$ , while  $H(x)$  is a sum of the "kinetic" part  $H_{\text{KIN}}(x; g, p]$ , which is an ultralocal quadratic form in  $p^{ab}(x)$ , and the "potential" part  $H_{\text{POT}}(x; g]$ , which is a scalar concomitant of  $g_{ab}(x)$  of the second order. This enables one to prove first that  $u_{ab}(x; g]$  must be local, i.e., a tensor concomitant of  $g$  of a finite order  $N$ , and then, by a glissade argument decreasing the order  $N$  step by step, end with the conclusion that  $u_{ab}(x; g]$  must actually be ultralocal. The only ultralocal tensor concomitant of  $g$  is however (a constant multiple of)  $g$  itself, and hence the only candidate for the dynamical variable (9.15) is the integrated trace of  $p$ :

$$E = - \int_{\Sigma} d^3x g_{ab}(x) p^{ab}(x) . \quad (9.17)$$

The Poisson bracket of  $E$  with the kinetic part  $H_{\text{KIN}}(x)$  of the super-Hamiltonian is proportional to  $H_{\text{KIN}}(x)$ :

$$\{E, H_{\text{KIN}}(x)\} = - \frac{3}{2} H_{\text{KIN}}(x) . \quad (9.18)$$

In other words, for each  $x \in \Sigma$ ,  $g_{ab}(x)$  is a conformal isometry (indeed, a homothetic motion) of the supermetric  $G_{abcd}(x; g)$ . Moreover,  $g_{ab}(x)$  is a "timelike" supervector, i.e.,  $G^{abcd}(x)g_{ab}(x)g_{cd}(x) < 0$ . Unfortunately,  $E$  does not scale the potential term  $H_{\text{POT}}(x; g]$  in the super-Hamiltonian at the same rate as the kinetic term  $H_{\text{KIN}}(x; g, p]$ :

$$\{E, H_{\text{POT}}(x)\} = \frac{1}{2} H_{\text{POT}}(x) . \quad (9.19)$$

The mismatch of the scaling factors in Eqs. (9.18) and (9.19) means that  $E$  does not weakly commute with the total super-Hamiltonian  $H(x)$ , and hence our search for an isometry of the geometrodynamical constraint system fails. As a result, one cannot make the product (9.14) positive definite by restricting the solutions of the Wheeler-DeWitt equation to those with a positive "energy" (9.15). The situation in asymptotically flat (rather than spatially compact) spacetimes is discussed by Friedman and Higuchi.<sup>161</sup>

- **Potential Term is not Positive.** Even if the super-Hamiltonian were stationary, one would still be unable to prove that the Klein-Gordon product (9.14) restricted to the states  $\mathcal{S}_0^+$  is positive definite because the potential  $|g|^{1/2} R(x; g]$  can be both positive and negative. I have discussed this difference between geometrodynamics and the relativistic particle in Section 3.

- **States with Negative E are Physical.** A relativistic particle moves in



*spacetime* along a future-oriented path, but a compact geometry moving in *superspace* can both expand and contract its total volume. The classical solutions of Einstein's equations can thus have either sign of  $E$ . The restriction of the quantum states to  $\mathcal{S}_0^+$  would simply eliminate one half of entirely permissible states.

- **Spacetime Problem.** The Klein-Gordon product (9.5) is the same on any spacelike hypersurface  $T(Q)$  in spacetime; similarly, the DeWitt product (9.14) is the same on any "spacelike" hypersurface  $T(x; g]$  in superspace. I have already pointed out that the intrinsic time  $T(x; g]$  cannot be interpreted as labeling the spacetime events (see the discussion of the spacetime problem in Section 5). One should also remember that the signature of the supermetric has nothing to do with the signature of the spacetime metric that is obtained by solving the classical Hamilton equations, and remains the same even in a Euclidean spacetime.<sup>46,162</sup> It is thus rather mysterious how the conservation of the DeWitt product in superspace is related to its conservation in spacetime.

Taken together, all of these points make it rather difficult to imagine how the space of solutions of the Wheeler-DeWitt equation can ever be turned into a Hilbert space.

## 10. Semiclassical Interpretation

Like the Klein-Gordon interpretation, the semiclassical interpretation tries to provide a dynamical interpretation of the Wheeler-DeWitt equation. It contends, however, that not necessarily all solutions of the Wheeler-DeWitt equation allow such an interpretation, but that time and quantum dynamics emerge only if a state  $\Psi$  becomes semiclassical. The semiclassical *interpretation* is thus imposed only after the Wheeler-DeWitt equation has been subjected to a semiclassical *approximation*.<sup>163</sup>

The semiclassical approximation was developed to bridge the gap between the ill-defined quantum theory of gravity and the much better understood quantum field theory on a given background. In its simplest form, it represents a heuristic derivation of the latter theory from canonical quantum gravity. In this process, a functional *Schrödinger* equation for the field emerges from the *Wheeler-DeWitt* equation for the field coupled to gravity. It is this recovery of the Schrödinger equation from the second-order Wheeler-DeWitt equation which underlies the semiclassical interpretation.

- **The Particle Model.** The semiclassical approximation is best explained on a finite-dimensional model (3.9) in which a heavy particle  $M$  interacting with

a light particle  $m$  is constrained to a stationary state.<sup>164</sup> The heavy particle is modeling the metric field and the light particle the matter field. For this purpose, it is suitable to assume that the metric  $G^{AB}$ , like the DeWitt supermetric (1.2), is indefinite, while the metric  $h^{kl}$  and the potential  $v$ , like the kinetic and potential energies of a field which satisfies the weak energy condition, are positive definite. In this case, one is allowed to put the total energy  $E$  equal to zero.

I adopt the Laplace-Beltrami factor ordering of the super-Hamiltonian  $H$ .

$$\hat{H} = -\frac{1}{2M} |Gh|^{-1/2} \partial_A |Gh|^{1/2} G^{AB} \partial_B + \frac{1}{2} MV - \frac{1}{2m} |h|^{-1/2} \partial_k |h|^{1/2} h^{kl} \partial_l + \frac{1}{2} mv. \quad (10.1)$$

The operator (10.1) is covariant under arbitrary transformations of the coordinates  $(Q^A, q^k)$ . Under this factor ordering,

$$\hat{H} = -\frac{1}{2M} (|h|^{-1/2} \partial_A |h|^{1/2}) G^{AB} \partial_B + \hat{H}_M + \hat{h}_m, \quad (10.2)$$

where

$$\hat{H}_M := -\frac{1}{2M} \Delta_G + \frac{1}{2} MV \quad \text{and} \quad \hat{h}_m := -\frac{1}{2m} \Delta_h + \frac{1}{2} mv. \quad (10.3)$$

The first term in Eq. (10.2) arises because the volume measure  $|h|^{1/2}$  depends on the position  $Q$  of the heavy particle.

The semiclassical approximation arises when one tries to solve the super-Hamiltonian constraint

$$\hat{H} \Psi(Q, q) = 0 \quad (10.4)$$

by expanding the state function  $\Psi(Q, q)$  in the inverse powers of the mass parameter  $M$ :

$$\Psi(Q, q) = e^{iMS(Q, q)} \phi(Q) \left( \psi(Q, q) + \sum_{n=1}^{\infty} \psi_{(n)}(Q, q) M^{-n} \right). \quad (10.5)$$

In relativity, the ansatz (10.5) corresponds to the weak-coupling expansion in powers of the Einstein constant of gravitation  $\kappa$ .

The leading term in  $\hat{H}\Psi$  is proportional to  $M^2$ . In this order, Eq. (10.4) tells us that  $h^{kl} S_{,k} S_{,l} = 0$ . Because the metric  $h^{kl}$  is positive definite,  $S$  cannot depend on the coordinates  $q^k$  of the light particle:  $S=S(Q)$ . In the  $M^1$  order, the

constraint (10.4) subjects  $S(Q)$  to the Hamilton-Jacobi equation

$$G^{AB}(Q)S_{,A}(Q)S_{,B}(Q) + V(Q) = 0 . \quad (10.6)$$

In the  $M^0$  order, one gets an equation for  $\Phi(Q)\psi(Q,q)$ . The split between the prefactor  $\Phi(Q)$  and the state function  $\psi(Q,q)$  is arbitrary. One can use this to simplify the equation for  $\psi(Q,q)$  by choosing a prefactor  $\Phi(Q)$  which satisfies the condition

$$\partial_A(|G|^{1/2}\Phi^2S^A) = 0, \quad S^A := G^{AB}S_{,B} . \quad (10.7)$$

The  $M^0$ -order equation then becomes

$$i|\hbar|^{-1}S^A\partial_A(|\hbar|^{1/2}\psi) = h_m(Q; \hat{q}, \hat{p}) \psi . \quad (10.8)$$

Equations (10.6) - (10.8) constitute the semiclassical approximation. In the next order of the expansion, i.e., in the  $M^{-1}$  order, one gets an equation for  $\psi_{(1)}(Q,q)$  which is of the form (10.8) except - and this is an important difference - the right-hand side of Eq. (10.8) is complemented by an inhomogeneous term constructed from the previously determined functions  $S(Q)$ ,  $\Phi(Q)$ , and  $\psi(Q,q)$ . This is the first term in the expansion which contains the second derivatives with respect to  $Q^A$ .

A particular solution  $S(Q)$  of the Hamilton-Jacobi equation (10.6) defines a congruence of classical trajectories.<sup>7</sup> These are the worldlines orthogonal to the hypersurfaces of constant  $S(Q)$ . I shall label these worldlines by  $N - 1$  "comoving" coordinates  $y^A(Q)$ ,

$$S^A(Q)y_{,A}^a(Q) = 0. \quad (10.9)$$

The solution  $S(Q)$  also defines a time function  $t(Q)$ : One chooses an arbitrary hypersurface transverse to the classical trajectories as a  $t = 0$  hypersurface. By Lie dragging this hypersurface along the vector field  $S^A(Q)$  one generates a foliation. The time function labeling the leaves of this foliation satisfies the condition

$$S^A(Q)t_{,A}(Q) = 1, \quad t_A := t_{,A} . \quad (10.10)$$

There are as many time functions as there are choices of the initial hypersurface. A comparison of Eqs. (10.10) and (10.6) reveals that for a general  $V(Q)$ , the time foliation necessarily differs from the  $S(Q) = \text{const}$  foliation.

In the coordinate system

$$Q^{A'} = (t(Q), y^a(Q)) \quad (10.11)$$

the remaining equations (10.7) and (10.8) simplify. The continuity equation (10.7) for the prefactor  $\bar{\Phi}(Q)$  takes the form

$$\partial_t ( |G(t,y)|^{1/2} \bar{\Phi}^2(t,y) ) = 0 . \quad (10.12)$$

This equation has the solution

$$\bar{\Phi}(t,y) = \varphi(y) |G(t,y)|^{-1/4}, \quad (10.13)$$

where  $\varphi(y)$  is an arbitrary function of the comoving coordinates. One can choose  $\varphi(y)$  such that the prefactor vanishes outside a thin worldtube centered on a selected classical worldline.

The equation (10.8) for the state function  $\psi$  reduces to

$$i |h(y; t,q)|^{-1/4} \partial_t ( |h(y; t,q)|^{1/4} \psi(y; t,q) ) = h_m(y; t, \hat{q}, \hat{p}) \psi(y; t,q) . \quad (10.14)$$

This is an  $(N-1)$  - parameter set of Schrödinger equations, one equation for each classical trajectory  $y$ . The Hamiltonian  $\hat{h}_m$  differs from one classical trajectory to another, and it depends on the time  $t$ , because the metric  $h^{kl}(Q,q)$  and the potential  $v(Q,q)$  depend in general on  $Q$ . Under the factor ordering (10.3), each Hamiltonian  $\hat{h}_m(y)$  is self-adjoint under the inner product

$$(\psi_1 | \psi_2) = \int d^n q |h(y; t,q)|^{1/2} \psi_1^*(y; t,q) \psi_2(y; t,q). \quad (10.15)$$

Moreover, this inner product does not depend on  $t$  by virtue of the Schrödinger equation (10.14).

In this scheme, the heavy particle follows a classical trajectory from the congruence determined by a fixed solution of the Hamilton-Jacobi equation (10.6), unaffected in its motion by the presence of the light particle. For each such trajectory, the motion of the light particle is quantized and described by the Schrödinger equation (10.14). This mixture of the classical and quantum worlds gave the semiclassical approximation its name.

▪ **The Klein-Gordon Product and the Schrödinger Product.** The Schrödinger equation (10.14) is a consequence of the Klein-Gordon constraints (10.4); accordingly, the Schrödinger norm (10.15) is induced by the Klein-Gordon

product (9.5).<sup>165</sup>

For the block-diagonal metric of the two-particle system (10.1), the current (9.3) constructed from a solution  $\Psi$  of the constraint (10.4) has the components

$$J^A = \frac{1}{2i} \frac{1}{M} G^{AB} (\Psi^* \overset{\leftrightarrow}{\partial}_B \Psi), \quad J^k = \frac{1}{2i} \frac{1}{m} h^{kl} (\Psi^* \overset{\leftrightarrow}{\partial}_l \Psi), \quad (10.16)$$

and it satisfies the continuity equation

$$|Gh|^{-1/2} \partial_A (|Gh|^{1/2} J^A) + |h|^{-1/2} \partial_k (|h|^{1/2} J^k) = 0. \quad (10.17)$$

For the state function (10.5), the current (10.16) in the lowest,  $M^0$ , order of the  $M^{-n}$  expansion takes the form

$$j^A = |\Phi(Q)|^2 S^A(Q) |\Psi(Q, q)|^2, \quad j^k = \frac{1}{2i} \frac{1}{m} h^{kl} (\Psi^* \overset{\leftrightarrow}{\partial}_l \Psi). \quad (10.18)$$

Because the prefactor  $\Phi(Q)$  was chosen so that it satisfies its own continuity equation (10.7), the continuity equation (10.17) reduces in the lowest order to the equation

$$|h|^{-1/2} S^A \partial_A (|h|^{1/2} |\Psi|^2 S^A) + |h|^{-1/2} \partial_k (|h|^{1/2} j^k) = 0 \quad (10.19)$$

for the state function of the light particle. In the comoving coordinates (10.9) and with respect to the time function (10.10), the continuity equation for the prefactor assumes the form (10.12) and Eq. (10.19) reduces to

$$\partial_t (|h|^{1/2} |\Psi|^2) + \partial_k (|h|^{1/2} j^k) = 0 \quad (10.20)$$

Equation (10.20) guarantees the conservation of the Schrödinger inner product (10.15). The form (10.18) of the current leads to the relation

$$\langle \Psi | \Psi \rangle = \int d^{N-1}y |G|^{1/2} |\Phi|^2 \cdot (\Psi | \Psi) \quad (10.21)$$

between the Klein-Gordon product (9.5) and the Schrödinger product (10.15).

▪ **Geometrodynamics with Sources.** At least formally, canonical geometrodynamics with a field source can be subject to the same treatment as the particle model (Banks<sup>164</sup>). The gravitational field plays the role of the heavy particle and the source field of the light one. The expansion parameter  $M^{-1}$  is the

Einstein constant of gravitation  $2\kappa$ . The three-metric  $g_{ab}(x)$  replaces  $Q^A$  and DeWitt's supermetric  $G_{abcd}(x;g)$  replaces  $G^{AB}(Q)$ . The source field  $\phi(x)$  replaces  $q^k$ . In geometrodynamics, the metric  $g_{ab}(x)$  enters as a coordinate  $Q^A$  both into the construction of the metric  $h^{kl}(Q,q)$  and of the potential  $v(Q,q)$  in the field space. For this reason, I took pain to explain how the model works for a  $Q$ -dependent metric  $h^{kl}$ , the point which is disregarded in the literature.

The gravitational field is largely unaffected by the source field because the gravitational coupling is weak. One seeks the solution of the Wheeler-DeWitt equation in the form

$$\Psi[g] = e^{i(2\kappa)^{-1}S[g,\phi]} \Phi[g] \left( \psi[g,\phi] + \sum_{n=1}^{\infty} \psi_{(n)}[g,\phi] (2\kappa)^n \right) . \quad (10.22)$$

As for the model system, one learns that the phase  $S$  cannot depend on  $\phi$  and that it satisfies the vacuum Hamilton-Jacobi equation (1.8) - (1.9). One is again entitled to subject the prefactor to the continuity equation

$$\frac{\delta}{\delta g_{ab}(x)} \left( |G(x;g)|^{1/2} S_{ab}(x;g) \right) = 0, \quad S_{ab}(x;g) := G_{abcd}(x;g) \frac{\delta S[g]}{\delta g_{cd}(x)} . \quad (10.23)$$

It then follows that the state functional  $\psi[g,\phi]$  satisfies the first-order variational equation

$$i |h(x;g)|^{-1/2} S_{ab}(x;g) \frac{\delta}{\delta g_{ab}(x)} \left( |h(x;g)|^{1/2} \psi[g,\phi] \right) = h_m(x;g, \hat{\phi}, \hat{\pi}) \psi[g,\phi] . \quad (10.24)$$

As the Wheeler-DeWitt equation itself, the equation (10.24) is actually a system of  $\infty^3$  equations, one per each point of  $\Sigma$ . The Jacobi principal function  $S[g]$  again enables one to introduce the comoving coordinates  $Y^R(x;g)$ ,  $R = 1, \dots, 5$ , and the intrinsic time function  $T(x;g)$  which satisfies the equation

$$S_{ab}(x';g) \frac{\delta T(x;g)}{\delta g_{ab}(x')} = \delta(x,x') . \quad (10.25)$$

In the new variables  $T(x;g)$ ,  $Y(x;g)$  on superspace, Eq. (10.24) takes the form of a functional Schrödinger equation in the intrinsic time  $T(x;g)$ :

$$\begin{aligned} i |h(x; Y, T)|^{-1/4} \frac{\delta}{\delta T(x)} (|h(x; Y, T)|^{1/4} \psi[Y; T, \phi]) \\ = h_m(x; Y, T, \hat{\phi}, \hat{\pi}] \psi[Y; T, \phi] \end{aligned} \quad (10.26)$$

▪ **Back Reaction.** The semiclassical approximation has been modified to include the back reaction of the quantized source field on the evolution of the classical geometry by Brout et al.<sup>166-168</sup>, Hartle,<sup>169</sup> Halliwell,<sup>170</sup> and Singh and Padhanabhan.<sup>171,163</sup> The aim of this development was to understand to what extent the classical background  $\gamma_{\alpha\beta}(X)$  can be considered as being produced by the expectation value of the energy-momentum tensor of the quantized source field  $\phi(x)$  according to the Einstein law

$$G^{\alpha\beta}(X; \gamma] = \kappa \langle T^{\alpha\beta}(X; \gamma, \phi) \rangle . \quad (10.27)$$

Equation (10.27) was originally proposed by Møller<sup>172</sup> and Rosenfeld<sup>173</sup> as an *exact* equation of a theory in which the metric field is always classical and only the ordinary matter fields are quantized. It was always felt that quantum gravity should yield Eq. (10.27) in a Hartree-Fock approximation. The development which I mentioned was aimed at getting the canonical version of Eq. (10.27) in the semiclassical approximation. For this purpose, Eq. (10.12) for the prefactor was proposed to be modified by the expectation value of a source term. One can trace a feeling of uneasiness in the literature about this procedure, the uneasiness which I happen to share. It seems to me that the semiclassical approximation in gravity is essentially a weak-field expansion in the powers of  $\kappa$  which should include the influence of the source field on geometry in the  $\kappa^1$  order, rather than by tempering with the prefactor in the lower order. I shall thus largely concentrate on the issue of time in the simpler and less controversial framework provided by the Schrödinger equation (10.26) (or its model counterpart (10.14)) on the classical background uninfluenced by the quantum sources.

▪ **Decoherence.** It has been argued that the emergence of the classical behavior requires not only the WKB type of the wave function, but also the destruction of the interference terms (the decoherence) among possible classical alternatives. Such a destruction can be brought in by the interaction of the modes one is interested in with the modes of an "environment" which is subsequently traced out from the density operator. Decoherence in the semiclassical approximation to quantum gravity was discussed by Zeh,<sup>174,175</sup> Joos<sup>176</sup>, Kiefer,<sup>177</sup> and Halliwell.<sup>178</sup>

▪ **Semiclassical Interpretation of the Model.** After explaining the

formalism of semiclassical *approximation*, let me turn to the proposals which aim at using it as a tool for *interpreting* quantum gravity and, in particular, for resolving the problem of time. This approach can be traced back to DeWitt<sup>32</sup> and Misner<sup>49</sup>; it was elaborated by Lapchinski and Rubakov<sup>179</sup>, by Banks<sup>164</sup>, and most clearly stated by Vilenkin.<sup>165</sup> I explain it first on the particle model. The difficulties which emerge at this level indicate that semiclassical interpretation does not cope with the problems of time even in the simple transition from quantum gravity with sources to quantum field theory on a given classical background. Furthermore, the semiclassical interpretation must be modified before it can be applied to quantum gravity proper. I shall later discuss the additional problems brought in by such a modification.

The semiclassical interpretation of the particle model is based on Eqs. (10.6), (10.9) - (10.10), (10.14), (10.12) and (10.20), and (10.15), (10.21). A particular solution of the Hamilton-Jacobi equation (10.6) defines a congruence of classical trajectories of the heavy particle. Equation (10.9) labels the trajectories, and Eq. (10.10) introduces the time function of the congruence. In that time, the wave function  $\psi$  of the light particle satisfies the Schrödinger equation (10.14) along a fixed trajectory. The continuity equation (10.12) for the prefactor provides the measure in the congruence of classical trajectories. The continuity equation (10.20) yields the conserved norm (10.15) on which the probabilistic interpretation of the state of the light particle can be based. The statistical ingredients are summarized by Eq. (10.21) which relates the Klein-Gordon inner product to the Schrödinger inner product (10.15).

The semiclassical interpretation asserts that the Klein-Gordon constraint (10.4) has a probabilistic interpretation only if the system is in a semiclassical state (10.5). In that case,

$$|G(t,y)|^{1/2} \Phi^2(t,y) d^{N-1}y \cdot |h(y; t,q)|^{1/2} |\psi(y; t,q)|^2 d^nq \quad (10.28)$$

is the joint probability of finding at the time  $t$  the heavy particle moving along the worldline  $y = \text{const}$  in the cell  $d^{N-1}y$  and the light particle localized in the cell  $d^nq$  about  $q$ . This interpretation is sometimes corroborated by evaluating the Wigner function of the semiclassical state (10.5) (Halliwell<sup>180</sup>).

The semiclassical interpretation has many weaknesses:

- **Global Time Problem.** The construction of the time function  $t(Q)$  presupposes that there exists a hypersurface which is transverse to the congruence of the classical trajectories generated by the Jacobi principal function  $S(Q)$ .<sup>165,170</sup>



This begs the global time problem. The problem becomes even more severe when one tries to superimpose the semiclassical states with different principal functions.

- **Superposition Problem.** The probabilistic interpretation (10.28) was reached under the assumption that the total system  $(Q,q)$  is in a single semiclassical state (10.5). However, the linearity of the constraint (10.4) allows it to be in a superposition of two or more such states,

$$\Psi(q,Q) = \Phi_1(Q)e^{iMS_1(Q)}\psi_1(Q;q) + \Phi_2(Q)e^{iMS_2(Q)}\psi_2(Q;q) . \quad (10.29)$$

When one calculates the current (10.16) for the state (10.29) to the lowest order  $M^0$ , one obtains the interference terms:

$$j^\Lambda = |\Phi_1|^2 S_1^\Lambda |\psi_1|^2 + |\Phi_2|^2 S_2^\Lambda |\psi_2|^2 + \Phi_1 \Phi_2 \left( \text{Re}(\psi_1 \psi_2^*) \cos M(S_2 - S_1) + \text{Im}(\psi_1 \psi_2^*) \sin M(S_2 - S_1) \right) (S_1^\Lambda + S_2^\Lambda) . \quad (10.30)$$

The interference of states with the same  $\Phi(Q)$  and  $S(Q)$  but different  $\psi_1(q)$ ,  $\psi_2(q)$  is desirable: it corresponds to the quantum nature of the light particle. The interference of the WKB states of the heavy particle is a different matter. It raises the question whether the semiclassical interpretation can be consistently maintained. The following discussion explores the various facets of the superposition problem.

- **Complex Structure Problem.** The Klein-Gordon constraint does not involve imaginary coefficients and thus allows real solutions  $\Psi$ . The real and imaginary parts of a complex solution are not coupled by Eq. (10.4). On the other hand, the Schrödinger equation (10.14) contains the imaginary unit associated with the time derivative, and it thus couples the real and imaginary parts of  $\psi$  (Barbour and Smolin,<sup>97</sup> Barbour<sup>181</sup>). This highlights the fact that the Schrödinger equation cannot be possibly derived from the Klein-Gordon constraint without some further tacit assumption.

Indeed, the Schrödinger equation for  $\psi(Q;q)$  follows from the  $M^0$  order of the  $M^{-n}$  expansion only when the state  $\Psi$  is limited to a *single* WKB state (10.5). The derivation no longer holds for a *superposition* (10.29). Even if one assumes that  $S_1$  and  $S_2$  separately satisfy the Hamilton-Jacobi equation (10.6) (which does not *follow* from the  $M$  order of the expansion procedure, though it is *compatible* with it), and subject both prefactors  $\Phi_1$  and  $\Phi_2$  to the continuity equation (10.7), the  $M^0$  order equation imposes only a *single* condition

$$\begin{aligned} & ( -i S_1^A |h|^{-\frac{1}{2}} \partial_A (|h|^{\frac{1}{2}} \psi_1) + \hat{h}_m \psi_1 ) \bar{\phi}_1 e^{iMS_1} \\ & + ( -i S_2^A |h|^{-\frac{1}{2}} \partial_A (|h|^{\frac{1}{2}} \psi_2) + \hat{h}_m \psi_2 ) \bar{\phi}_2 e^{iMS_2} = 0 \end{aligned} \quad (10.31)$$

on *two* state functions  $\psi_1$  and  $\psi_2$ . Equation (10.31) has many more solutions than those in which  $\psi_1$  and  $\psi_2$  satisfy the separate Schrödinger equations. For example, one can put  $\psi_1 = \psi_2 =: \psi$  and derive an equation for  $\psi$ ,

$$\begin{aligned} & i ( \bar{\phi}_1 e^{iMS_1} + \bar{\phi}_2 e^{iMS_2} )^{-1} ( \bar{\phi}_1 e^{iMS_1} S_1^A + \bar{\phi}_2 e^{iMS_2} S_2^A ) \\ & |h|^{-\frac{1}{2}} \partial_A (|h|^{\frac{1}{2}} \psi) = \hat{h}_m \psi , \end{aligned} \quad (10.32)$$

which is *not* a Schrödinger equation, because the vector field along which the directional derivative is taken is complex. In particular, for  $S_1 = -S_2 =: S$ , and  $\bar{\phi}_1 = \bar{\phi}_2$ , one gets the equation

$$- \tan MS \cdot |h|^{-\frac{1}{2}} \partial_t (|h|^{\frac{1}{2}} \psi) = \hat{h}_m \psi \quad (10.33)$$

whose coefficients are all real. This clearly shows that the complex structure which one associates with the Schrödinger equation, and the equation itself, do not follow from the Klein-Gordon constraint by the  $M^{-n}$  expansion. An elucidating discussion of simple state functions of the type (10.33) is given in Barbour and Smolin.<sup>97</sup>

• **The Double Standard of Time.** Even those superpositions (10.29) in which the coefficients  $\psi_1$  and  $\psi_2$  separately satisfy the Schrödinger equations present problems. The Jacobi principal functions  $S_1(Q)$  and  $S_2(Q)$  lead to different time functions  $t_1(Q)$  and  $t_2(Q)$  and to different comoving coordinates  $y_1^a(Q)$  and  $y_2^a(Q)$ . No single choice of the coordinate system will simultaneously reduce the first-order equations (10.8) to the Schrödinger form (10.14) and the continuity equations (10.7), (10.19) to the form (10.12), (10.20). On what time function should then one base the probabilistic interpretation of the pure state (10.29)?

One can counter this objection by pointing out that one does not need to commit oneself to any particular time function when evaluating the norm. The exact current (10.16) is exactly conserved, and the inner product (9.5) is thus the same on every hypersurface. When one replaces the exact current by the semiclassical current, the inner product is still the same on every hypersurface, up to the terms of order  $M^{-1}$ . This suggests that the integrand of

$$\langle \Psi | \Psi \rangle = \int d^{N-1}y \int d^nq \quad t_A j^A |G|^{1/2} |h|^{1/2} \quad (10.34)$$

can be interpreted as the probability density  $\rho$  to find the heavy particle in the cell  $d^{N-1}y$  and the light particle in the cell  $d^nq$  on *any* hypersurface  $t(Q) = \text{const}$ .

Some difficulties, however, still remain. One is not entitled to say that the heavy particle is in a definite state of motion. The global problem of time is aggravated because there may not exist any hypersurface which is *simultaneously* transverse to both congruences of classical trajectories. Most importantly, the probability density  $\rho$  does not need to be positive.

- **Interference Problem.** The interference terms in Eq. (10.30) indicate that it may not be possible to say that the heavy particle is in a definite state of motion. One can try to exorcise them by one of the two alternative methods:

I) One can choose the two prefactors  $\Phi_1$  and  $\Phi_2$  so that they do not overlap, concentrating, e.g.,  $\Phi_1$  in a thin tube surrounding a classical trajectory  $y_1$ , and  $\Phi_2$  in a thin tube surrounding a classical trajectory  $y_2$ . As long as the two tubes do not intersect,

$$j^A = |\Phi_1|^2 |\psi_1|^2 S_1^A + |\Phi_2|^2 |\psi_2|^2 S_2^A \quad (10.35)$$

This form of the current corresponds to a mixed state. This simultaneously solves the problem with the double standard of time, by using a time function  $t$  which coincides with  $t_1$  along the tube  $y_1$ , and with  $t_2$  along the tube  $y_2$ . However, why should nature select for us the prefactors which kill the interference terms?

II) One can argue with Vilenkin<sup>165</sup> that for  $S_1 \neq S_2$  and for a large  $M$ , the  $\sin M(S_2 - S_1)$  and  $\cos M(S_2 - S_1)$  are rapidly oscillating functions on a transverse hypersurface which, in the limit  $M \rightarrow \infty$ , average out to zero when integrated against the slowly varying functions  $\Phi_1 \Phi_2$  and  $\psi_1 \psi_2^*$ . However,  $M$ , though large, is actually finite, and narrow interference fringes still remain, similar to the interference fringes of a massive particle in the double-slit experiment. Given a sufficiently fine resolution, the pure state (10.30) can be distinguished from the mixed state (10.35).

- **Negative Norms.** The greatest puzzle surrounding the semiclassical interpretation is how it can avoid the negative probabilities inherent in the Klein-Gordon product. This puzzle is highlighted by Eq. (10.21) whose left-hand side is indefinite and the right-hand side is positive definite.

A closer scrutiny reveals the trick. The semiclassical interpretation

achieves the positivity in two steps: 1) by restricting the space of solutions of the Klein-Gordon equation to the states of the form (10.5) with a *single* Jacobi principal function  $S(Q)$ , and 2) by choosing a hypersurface  $t(Q) = 0$  transverse to the classical trajectories of  $S(Q)$  and orienting it in the direction of the vector field  $S^A(Q) : t_A(Q)S^A(Q) > 0$ .

The positivity of the norm is thus bought at the price of a serious infringement of the superposition principle. It is hard to see what physical mechanism selects only the states (10.5) with a single  $S(Q)$  and rejects all other possible solutions of the Klein-Gordon constraint. Characteristically, the discussion of the semiclassical interpretation of the state function of the Universe is often connected with cosmological speculation about the uniqueness of this state (Banks,<sup>164</sup> Vilenkin<sup>182</sup>). (An interpretation of an "arbitrary" single state was proposed by Padhanabhan<sup>183</sup> and Greensite<sup>184</sup>.)

If one accepts an argument which removes the interference terms from Eq. (10.30), some superpositions (10.29) become permissible. However, one can never include all the states and still maintain the positivity. This is especially clear when one tries to superimpose classical states with the Jacobi functions  $S$  and  $-S$ .

If the Hamilton-Jacobi equation (10.6) has a solution  $S(Q)$ , it necessarily also has a solution  $-S(Q)$ . For a relativistic particle, one of those solutions may be excluded by the requirement that the particle must have a positive energy. No such requirement exists in vacuum geometrodynamics, where both  $S[g]$  and  $-S[g]$  are allowed, and stand to each other as an expanding versus a contracting Universe. I base the discussion on the time function (10.10) oriented in the direction of  $S^A(Q)$ .

It is easy to give examples of states with zero norm. The real state  $\Psi = \Phi(e^{iMS} + e^{-iMS})\psi$ , where  $\psi$  solves the real equation (10.33) instead of the Schrödinger equation, is such a state. The real state  $\Psi = \Phi\psi e^{iMS} + \Phi\psi^* e^{-iMS}$ , where  $\psi$  solves the Schrödinger equation associated with  $S$  and hence  $\psi^*$  solves the time-reversed Schrödinger equation associated with  $-S$ , is another such state. The state  $\Phi\psi e^{iMS}$  taken by itself has a positive norm, the state  $\Phi\psi^* e^{-iMS}$  has a negative norm. As long as the probabilistic interpretation of semiclassical theory is derived from the Klein-Gordon product, as in Eq. (10.21), one cannot allow the Hilbert space to be spanned by the semiclassical states corresponding to *all* possible Jacobi principal functions. At least one half of such states must be excluded. The semiclassical interpretation by itself does not give any hint what collection of states spans the Hilbert space and what states are to be left out.

- **Higher Corrections Problem.** The expectation that the semiclassical approximation solves the problem of time was raised by the emergence of the first-order Schrödinger equation from the second-order Klein-Gordon equation.

As pointed out by Barbour and Smolin,<sup>97</sup> the Schrödinger equation does not automatically follow in the  $M^0$  order, and there are solutions to this order which satisfy an alternative first-order equation, Eq. (10.32). In the  $M^{-1}$  order, the situation gets worse. I pointed out that the equation for the first-order correction  $\psi_{(1)}$  to the state function  $\psi$  contains an inhomogeneous term. This term appears because the Klein-Gordon constraint is a second-order, rather than the first-order, differential equation. The Schrödinger equation does not survive to the  $M^{-1}$  order. The quantum gravitational corrections to the Schrödinger equation for matter fields were discussed by Kiefer and Singh.<sup>185</sup>

- **Back Reaction and the Problem of Time.** The back reaction term was included into semiclassical approximation by amending the equation for the prefactor by the expectation value of the source term. Irrespective of whether such a procedure is or is not justified, it only further complicates the issue of time. It replaces the Schrödinger equation (10.26), or its model counterpart (10.14), by an integro-differential equation for the state  $\psi$ . This equation is nonlinear, and hence does not satisfy the superposition principle. The difficulties associated with such an equation were analyzed by several authors.<sup>186-191</sup>

- **Semiclassical Interpretation of Quantum Gravity.** The interpretation I have discussed so far is not directly applicable to quantum gravity. To maintain that quantum gravity has a probabilistic interpretation only if the Universe is in a semiclassical state (10.22) would literally mean that *quantum* gravity has a probabilistic interpretation only if it is *classical*. Obviously, the literal interpretation must be amended. This is achieved by the qualification that not all gravitational variables need to be classical; only those which represent an internal clock must become classical, while the true gravitational degrees of freedom should stay quantized. Unfortunately, this requires a split of the gravitational variables into those which represent time and become classical, and those which represent dynamics and remain quantum. Such a split carries to the semiclassical interpretation all the problems which beset the internal Schrödinger interpretation. I shall first discuss two major attempts to introduce a split into semiclassical interpretation, and then point out their difficulties.

- **Inhomogeneous Perturbations of a Homogeneous Universe.** The first attempt (Halliwell and Hawking<sup>48</sup>) assumes that the Universe is nearly homogeneous. Small inhomogeneities propagate on the homogeneous background and influence the evolution of the homogeneous mode, as in the fourth model of Section 3. Halliwell<sup>170</sup> then assumes that the homogeneous mode is in the WKB state while the inhomogeneous perturbations are quantized. The time variable is associated only with the homogeneous mode.

▪ **Quantum Perturbations of a Classical Background.** The second attempt (Vilenkin<sup>165</sup>) splits the quantum part  $q_{ab}(x)$  of the spatial metric from the classical part  $Q_{ab}(x)$  point by point, the quantum part being multiplied by the small parameter  $\kappa$  :

$$g_{ab}(x) = Q_{ab}(x) + 2\kappa q_{ab}(x) . \quad (10.36)$$

As one expands the super-Hamiltonian (1.2) with respect to  $2\kappa$ , the quantum part  $q_{ab}(x)$  and its conjugate momentum enter the expanded constraint in the same order (namely,  $(2\kappa)^0$ ) as the source-field terms. They can thus formally be grouped with the field terms in the semiclassical expansion (10.22) of the state functional  $\Psi$ . Vilenkin talks about the  $q_{ab}(x)$  variables as "corresponding to a small subsystem of the Universe" and justifies the split (10.36) by saying that "the semiclassical character of the Universe and the smallness of the subsystem are both due to the fact that the Universe is large."

• **Quantum Gravity and Time in the Small.** The Halliwell proposal, unlike the Vilenkin proposal, reduces the many-fingered time Schrödinger equation (10.26) to an ordinary Schrödinger equation in the "cosmological time" associated with the homogeneous mode. The privileged foliation labeled by this single time variable is specified by the assumption that the geometry is "almost homogeneous" on its leaves. As in linearized gravity, this still allows a small flexibility to wiggle the leaves. To fix the leaves completely, one must impose some "gauge conditions" on the inhomogeneous modes.

Such a specification of what is meant by time in quantum gravity would be justifiable if the Universe were almost homogeneous down to the scales at which one expects quantum gravity to become significant, i.e., down to the scales comparable with the Planck length. This is blatantly not true. The Universe is inhomogeneous even on much longer scales. To specify the foliation in a highly inhomogeneous Universe by the requirement that its leaves are almost homogeneous makes no sense, and Halliwell's method of choosing the mode which remains classical is no longer applicable. Halliwell's method remains what it probably was intended to be from the very beginning, namely, an interesting model, but it does not solve the problem of time in semiclassical interpretation of quantum gravity.

The proponents of the semiclassical interpretation may retort that in an inhomogeneous Universe the foliation should be entirely fixed by a slicing ("coordinate") condition. I criticized such a draconian solution in connection with the problem of functional evolution (Section 4). Moreover, because so far no slicing condition was found which would be privileged over the others, one must face the multiple choice problem.

- **Perturbative Canonical Quantization?** It is not at all clear in what sense the classical part of the split (10.22) corresponds to the Universe in the large and the quantum part to its small subsystem; intuitively, the behavior of the Universe in the large should be associated with some collective, integrated, variables (analogous to the center of mass of a system of particles) and its behavior in the small with the deviation from the collective variables. On the other hand, the split (10.22) is purely local, made point by point in  $\Sigma$ , and the "smallness" is brought in by the formal insertion of a small parameter  $\kappa$ . There is no physical principle which would unambiguously determine the split of small quantum perturbations from a large classical background. This suggests that either the whole metric field is to be treated classically, or the whole of it should be quantized (Singh and Padhanabhan<sup>163</sup>). In the first case, there is no quantum gravity, and in the second case, there is no semiclassical interpretation. The formal split (10.21) attempts at the canonical level what perturbative "covariant" quantization tried to do at the spacetime level. The failures of the perturbative covariant quantization are well known.

- **Classical Modes and Decoherence.** Even if one could defend a particular split, the question still remains why variables on one side of this split become classical while variables on the other side of the split remain quantum. At this point, one often hears a decoherence argument: A particular set of variables becomes classical because their interaction with the environment makes them to decohere. This type of argument was invoked by Kiefer<sup>177</sup> as an explanation why the homogeneous mode in the Halliwell-Hawking split becomes classical. In particular, Kiefer argued that the scale factor ("the radius") of the Friedmann universe driven by a massive scalar field is rendered classical by its interaction with inhomogeneous modes. The trouble with this kind of argument is that there seems to be no specific input which would make it to work for a given mode and to fail for other modes. One would thus expect that in a typical nonlinear field theory, when one chooses some higher, inhomogeneous mode, and treats all the other modes as an "environment", the interaction of the environment would again lead to decoherence. The decoherence arguments in their present form thus seem to be ill suited to provide a selection principle which determines the split of the classical modes from the quantum ones. A similar objection can be raised against Halliwell's argument<sup>178</sup> that the expanding and contracting modes in the state function of the universe decohere. It is thus difficult to dispute away the issue of the negative norms.

- **Spacetime Problem.** As in the Klein-Gordon interpretation, the many-fingered time variable (10.25) of the semiclassical interpretation is to be constructed from the spatial metric. No spacetime scalar can be built in this

way.

In summary, the semiclassical interpretation does not solve the standard problems of time. It merely obscures them by the approximation procedure and, on the way, creates more problems.

## 11. Third Quantization

For a relativistic particle moving on a dynamical background, the failure to define the positive-energy sector of the solution space is taken as an indication that the one-particle interpretation of the theory cannot be consistently maintained because particles are produced and destroyed by the changing metric and/or the changing rest mass of the particle. This suggests that the quantum theory of a relativistic particle on a dynamical background makes sense only on an extended Hilbert space incorporating the states with an arbitrary number of particles. Formally, this is achieved by *second quantization*, in which the wave functions  $\psi$  satisfying the Klein-Gordon constraint are turned into quantum field operators. These operators can be thought of as acting on the state functionals  $\Psi[\psi]$  of the fields. At least in simple cases, this representation is equivalent to the Fock space representation. Such a standpoint is, of course, experimentally corroborated in a closely related situation, namely, when the particle is an electron and the changing background is realized by external electromagnetic potentials.

The geometry  $g$  moving in superspace is analogous to a relativistic particle moving on a dynamical background. The role of the background is played by DeWitt's supermetric  $G_{abcd}[g]$  and by the scalar curvature term  $-|g|^{1/2}R[g]$ ; the dynamical nature of this background is demonstrated by a theorem proved by Kuchař.<sup>159</sup> By pursuing the analogy with a relativistic particle, one can surmise that the dynamical background in superspace produces geometries in the same way that the dynamical background in spacetime produces particles. If so, the one-geometry interpretation of quantum geometrodynamics cannot be consistently maintained and one should work in an extended Hilbert space incorporating many-geometry states. Heuristically, one should turn the state functionals  $\Psi[g]$  which satisfy the Wheeler-DeWitt equation into quantum fields  $\hat{\Psi}[g]$  on superspace that act on some higher-order functionals  $\tilde{\Psi}[\Psi[g]]$ .

A state functional  $\Psi[g]$  which satisfies the Wheeler-DeWitt equation already describes a state of the quantum gravitational *field*, i.e., of a second-quantized zero mass relativistic particle of spin 2. (Similarly, a state functional  $\Psi[X,B]$  that satisfies the constraint equations of parametrized Maxwell theory describes a state of the quantum electromagnetic *field*, i.e., of a second-quantized zero mass relativistic particle of spin 1.) The process in which the state functional  $\Psi[g]$  is turned into an operator thus deserves to be called *third quantization*. The



logical possibility of third quantization was noticed by Kuchar<sup>159</sup> and taken as a serious physical option by Giddings and Strominger,<sup>192</sup> Coleman,<sup>193</sup> McGuigan,<sup>194,195</sup> and others.

The question which interests me here is whether third quantization can help to resolve the problems of time in quantum gravity. Unfortunately, very little work has been devoted to this question. The papers on the subject have developed the formalism rather than consider its interpretation. In this state of affairs, I cannot do much more than offer some rather general remarks about the conceptual difficulties of third quantization as an interpretational framework.

• **Does Third Quantization Solve the Problem of Time?** The expectation that third quantization solves the problem of time in quantum gravity is based on a belief that the second quantization solves the problem of time in quantum theory of a relativistic particle. Before going any further, it is advisable to check whether this belief is substantiated.

The second quantization approach to a quantum field theory is based on the construction of a Fock space. To build the Fock space, one takes a one-particle Hilbert space  $\mathcal{S}_{(1)}$ . From the direct product of the one-particle states  $|\psi\rangle \in \mathcal{S}_{(1)}$ , one builds the states

$$|\Psi_N\rangle = S |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle \quad (11.1)$$

which span the N-particle sector  $\mathcal{S}_{(N)}$ . Symmetrization S under exchange of the particles is appropriate for Bose statistics. The Fock space  $\mathcal{F}$  is the direct sum of such sectors,

$$\mathcal{F} = \mathcal{S}_{(0)} \oplus \mathcal{S}_{(1)} \oplus \mathcal{S}_{(2)} \oplus \cdots \quad (11.2)$$

Here,  $\mathcal{S}_{(0)}$  is spanned by the vacuum state  $|\Psi_0\rangle$ .

The norm of the special N-particle state (11.1) is induced by the norm of its one-particle constituents:

$$\langle \Psi_N | \Psi_N \rangle := \prod_{i=1}^N \langle \psi_i | \psi_i \rangle \quad (11.3)$$

The norm of a state

$$|\Psi\rangle = |\Psi_0\rangle \oplus |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots \quad (11.4)$$

in  $\mathcal{F}$  is the sum

$$\langle \Psi | \Psi \rangle = \sum_{N=0}^{\infty} \langle \Psi_N | \Psi_N \rangle \quad (11.5)$$

of the norms of its N-particle components.

This short description makes it clear that one can define the Fock space  $\mathcal{F}$  only if one knows the one-particle space  $\mathcal{F}_{(1)}$  as a Hilbert space. This brings us back to the Hilbert space problem for a relativistic particle.

If the particle moves on a stationary background, the one-particle Hilbert space is spanned by the positive-energy solutions of the Klein-Gordon constraint (9.1) - (9.2), and the inner product is given by Eq. (9.5). If the particle moves on a dynamical background, it is not clear what one should take for the one-particle Hilbert space. If the background is stationary in the past or stationary in the future, one can construct the corresponding incoming and outgoing Hilbert spaces and answer scattering questions. However, one does not know how to pose meaningful questions in the dynamical region.

The absence of a privileged one-particle Hilbert space structure is the source of ambiguities in constructing a unique quantum field theory (even a free field theory) on a dynamical background. There are many ways of reformulating this problem, and in some of these the connection with the time problem (the Hilbert space problem) of the one-particle theory is far from obvious. It is then easy to start believing that the problem of time disappeared from the second-quantized theory and that one should merely fix some other technical problems. What I want to stress is that in one way of looking at things, the ambiguities of quantum field theory on a dynamical background are the direct consequence of the Hilbert space problem of the one-particle theory, and thus reflect the problem of time for the second-quantized system. The second quantization merely shifts the problem of time to a different level without really solving it. This is not a good omen for the attempts to solve the time problem in quantum gravity by third quantization.

- **How to Observe the Superfield?** Suppose that the ambiguities of a quantum field theory on a dynamical background were somehow resolved without having to solve the Hilbert space problem for the one-particle system. Let me further grant that the same scheme could be successfully applied to third-quantized gravity. How would one interpret such a theory?

The states of a quantum field can be given either in the *field representation* or in the *particle representation*. These representations reflect the choice of fundamental observables. The question is: How does one measure these observables?

The fundamental observables in the field representation are the spacetime field operators, the scalar field  $\hat{\psi}(X)$ , the electromagnetic field  $\hat{F}_{\alpha\beta}(X)$ , etc., or their canonical counterparts  $\hat{\psi}(x)$ ,  $\hat{\pi}(x)$ ;  $\hat{\mathbf{E}}(x)$ ,  $\hat{\mathbf{B}}(x)$ , etc. The measurability of the electric and magnetic field operators played a major role in clarifying the conceptual foundations of quantum field theories.<sup>121</sup> The construction of

appropriate thought experiments required a truly heroic effort.<sup>196</sup> I suspect that an equally heroic effort would be needed to understand how one should measure the scalar field  $\hat{\Psi}[g]$  on superspace. Indeed, the ordinary experimental devices operate in *spacetime* and are designed to measure spacetime quantities, like the average field strength in a spacetime region. Nobody as yet has imagined a piece of equipment which would be able to measure something like the mean value of the field  $\hat{\Psi}[g]$  in a region of superspace.

- **How Could a Many-Geometry State be Observed?** If it is difficult to imagine an apparatus which would measure the "field observable in superspace," perhaps it is easier to imagine an apparatus which would measure the "particle observable in superspace." A complete set of particle observables in spacetime is the number of particles and their positions at a given instant, i.e., on a given spacelike hypersurface. The apparatus which performs such a measurement is a collection of small Geiger counters densely distributed throughout space and brought into the state of readiness at a given instant of time. Notice that this is the *same* apparatus as that designed to measure the position of a *single* particle; the fact that the apparatus operates on a state in the Fock space rather than on a one-particle state manifests itself by more than one Geiger counter being triggered, i.e., in the *response* of the apparatus, rather than in its *construction*.

In third-quantized gravity, the many-particle states are replaced by many-geometry states. However, the analogy is not perfect. A three-geometry is not analogous to the position of a particle, because it contains also information about intrinsic time. This brings the problem of time back into the game. To be able to talk about measurements of the three-geometry rather than about measurements of a more complicated object (say, the conformal three-geometry), I can use the results of section 7, and couple gravity to a suitable material system, like a Gaussian reference fluid. (In a classical phenomenological discussion, one traditionally uses for the same purpose a system of small rigid rods dragged by the reference points,<sup>197</sup> or perfect clocks carried by the reference points together with the radar signals.<sup>198</sup>) It is fairly clear what response of such a material system considered as a classical apparatus corresponds to the detection of a definite three-geometry. If one takes the analogy with the Geiger counters seriously, the same material system should be able to detect also a many-geometry state. I must admit, however, that I am unable to imagine a classical response of the material systems like rods, clocks, signals, or a Gaussian reference fluid, which could be appropriately described by saying that on a single spacelike hypersurface one has detected two or more intrinsic geometries.

- **Changing Topology.** An alternative interpretation of the many-

geometry state may be that the individual geometries  $g_1$  and  $g_2$  are not carried by a single hypersurface, but by *different* copies of  $\Sigma$ . Isham discusses the difficulties of this version of the interpretation.<sup>96</sup> First, the transition from a one-geometry state to a two-geometry state corresponds now to a topology change in which  $\Sigma$  bifurcates into two copies of itself. It is doubtful whether such a change is compatible with the third-quantized Wheeler-DeWitt formalism. Second, the Bose structure of the Fock space loses its operational significance. The symmetry of the two-geometry state should reflect the indistinguishability of the individual geometries. However, if the two copies of  $\Sigma$  are disjoint, there is presumably no causal connection between them, and it does not make sense to compare the two geometries with each other. (In the same way, it would not make much sense to talk about two particles as being indistinguishable if each of them moved in a separate space rather than in the same space.)

In summary, it seems that the third quantization does not really solve the problem of time, and that it is extremely hard to interpret its formalism operationally.

## 12. Naive Schrödinger Interpretation

Of all the proposals on how to interpret quantum gravity, the naive Schrödinger interpretation is certainly the most straightforward. It asserts that the square

$$|\Psi[g]|^2 \tag{12.1}$$

of a solution  $\Psi[g]$  of the Wheeler-DeWitt equation (2.5) and the supermomentum constraint (2.4) is the probability density for finding a hypersurface with the intrinsic geometry  $g$ . The designation "naive" (coined by Wald and Unruh<sup>76</sup>) distinguishes it from the internal Schrödinger representation, which never speaks about probabilities before splitting the dynamical variables from the many-fingered time. By denying the need for such a split, the naive Schrödinger interpretation is the first scheme which maintains that one does not need to know what is time before interpreting quantum gravity.

The naive Schrödinger interpretation was proposed by Hawking<sup>199-203</sup> and widely used by the members of his school. The conditional interpretation and the sum-over-histories interpretation, which I consider in the following two sections, may be considered as refinements of the naive Schrödinger interpretation. Here I shall discuss the naive Schrödinger interpretation in its unadulterated (and thus intuitively appealing) form.

The conceptual meaning of the naive Schrödinger interpretation is clarified

when one applies it to a simple particle model. For a parametrized Newtonian or relativistic particle, the naive Schrödinger interpretation amounts to the statement that

$$|\Psi(T, Q)|^2 \quad (12.2)$$

is the probability density for finding an instant  $T$  with the particle in the position  $Q$ . The peculiar timeless nature of the interpretation is brought thereby into a sharp focus.

• **Naive Interpretation Lacks Dynamics.** A comparison of Eqs. (12.1) and (12.2) reveals that an attempt to normalize the state functional over the geometries  $g$  is analogous to an attempt to normalize the wave function of a particle over both the position and the time. Such an attempt must fail because, due to the Schrödinger or the Klein-Gordon equation, the wave function cannot be square integrable in the time direction. One is thus forced to talk only about relative probabilities.

I shall consider some typical probabilities which the interpretation (12.2) allows me to calculate for a Newtonian particle. I normalize  $\Psi(T, Q)$ , as in the standard quantum mechanics, to 1 at a given instant  $T$ :

$$\rho(T) := \int dQ |\Psi(T, Q)|^2 = 1. \quad (12.3)$$

The Schrödinger equation then implies that  $\rho(T) = 1$  for all  $T$ . The naive Schrödinger interpretation allows me to reinterpret the expression (12.3) as the *relative probability to find an instant  $T$* . One sees that in nonrelativistic quantum mechanics all instants are equally probable; to find one instant is as probable as to find any other. Similarly,

$$\rho(Q) := \lim_{T' \rightarrow \infty} \frac{1}{2T'} \int_{-T'}^{T'} dT |\Psi(T, Q)|^2 \quad (12.4)$$

is the *probability density to find the particle at  $Q$  irrespective of when*. Finally, I can write the expression

$$\langle Q \rangle := \lim_{T' \rightarrow \infty} \frac{1}{2T'} \int_{-T'}^{T'} dT \int dQ Q |\Psi(T, Q)|^2 \quad (12.5)$$

for the *mean position of the particle during the whole motion*.

One sees that the probabilistic scheme (12.3) - (12.5) is internally coherent. Thus, e.g., taking an ensemble of systems described by the state function  $\Psi$ , and measuring  $Q$  without measuring  $T$ , one is, according to Eq. (12.3), as likely to measure  $Q$  at one instant as at any other instant; by averaging

all such  $Q$ 's obtained at different instants  $T$ , one ends with the mean value formula (12.5).

However, one is missing the answer to a typical dynamical question which one can ask in the internal Schrödinger interpretation, namely "If one finds the particle at  $Q$  at the time  $T$ , what is the probability of finding it at  $Q'$  at the time  $T' > T$ ?" Such questions make sense and can be realized by suitable experimental arrangements. Thus, the naive Schrödinger interpretation simply fails to ask and answer interesting dynamical questions.

Returning back from the particle model to quantum gravity, one sees that the naive Schrödinger interpretation can pose some questions, like "What is the mean value of this functional of the intrinsic geometry?", but these are not dynamical questions. A typical application of the naive Schrödinger interpretation is "To find the probability of this or that Universe" (leading to statements like "A Universe with a given density of mass is very likely flat and big") but *not* "To find the probability of this or that *evolution* of the same Universe." One of the main defects of the naive Schrödinger interpretation is that it does not describe how the Universe is changing.

- **Inappropriate for the Wheeler-DeWitt Equation.** The naive Schrödinger interpretation for a Newtonian particle is compatible with the internal Schrödinger interpretation, as exemplified by Eqs. (12.3) - (12.5). This can no longer be maintained for a relativistic particle, even on a Minkowskian background. Because the Schrödinger norm is no longer conserved by the Klein-Gordon evolution, Eq. (12.3) breaks down, and different instants of the Minkowskian time are not equally probable. Further, the multiplication operator  $\hat{Q} = Q \times$  is not the position operator of a relativistic particle, and the expression (12.5) loses its meaning of the mean position. This shows that the naive Schrödinger interpretation is inappropriate for a relativistic particle. The Wheeler-DeWitt equation is certainly closer to the Klein-Gordon equation than it is to the Schrödinger equation. The naive Schrödinger probability (12.1) does not seem to be appropriate for it either.

- **Violation of the Constraints.** The interpretation of the expression (12.1) as the probability density of the intrinsic geometry  $g$  is consistent with the standard rules of quantum mechanics only if the geometry  $g$  is represented by the multiplication operator. However, the multiplication operator  $\hat{g}(x) = g(x) \times$  throws the state  $\Psi[g]$  out of the physical space of states which satisfy the Wheeler-DeWitt equation. In other words, a measurement of  $g(x)$  violates the constraints. The naive Schrödinger interpretation avoids this issue because it refrains from making any statement about what is the state function of the Universe after one has found a hypersurface with the geometry  $g(x)$ . If one

believes that the intrinsic geometry is a carrier of information about time,<sup>2</sup> one should recall the old arguments why time should not be represented by a self-adjoint operator in quantum mechanics.

### 13. The Conditional Probability Interpretation

The conditional probability interpretation is an attempt to introduce dynamics into the naive Schrödinger interpretation. This requires an identification of time. However, the advocates of the interpretation hold that it is unnecessary to cast the constraints into the Schrödinger form with respect to any specific choice of time. Instead, they work within the Wheeler-DeWitt framework and assert that they can handle all choices of time at once without running into the multiple choice problem. They also feel that the lack of global time function is not necessarily a problem. In these respects, the conditional probability interpretation substantially differs from the internal Schrödinger interpretation.

The conditional probability interpretation was developed by Page and Wootters.<sup>107,204-208</sup> I sidestep those aspects of their discussion which relate the interpretation to the unobservability of the label time and the superselection rule for the conjugate energy.

Like the naive Schrödinger interpretation, the conditional probability interpretation works with the Hilbert space  $\mathcal{F}$  of square-integrable functions on the configuration space(time) of the unconstrained system. The interpretation assigns probabilities to the measurements of the projection operators  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$  on  $\mathcal{F}$ . It is well known that any projection operator

$$\hat{A}^2 = \hat{A} \quad (13.1)$$

has only two eigenvalues, 0 and 1. The eigenfunctions  $|\phi_0\rangle$  and  $|\phi_1\rangle$  to those eigenvalues span two mutually orthogonal linear subspaces of  $\mathcal{F}$ ,  $\mathcal{F}_{A=0}$  and  $\mathcal{F}_{A=1}$ :  $\mathcal{F} = \mathcal{F}_{A=0} \oplus \mathcal{F}_{A=1}$ . The measurement of A amounts to answering the yes - no question "Does  $|\Psi\rangle$  lie in  $\mathcal{F}_{A=1}$ ?" The eigenvalue A=1 means that the answer is "Yes", the eigenvalue A=0 that the answer is "No."

Let  $|\Psi\rangle \in \mathcal{F}_0 \subset \mathcal{F}$  be a physical state, i.e., a solution of the super-Hamiltonian constraint

$$\hat{H}|\Psi\rangle = 0. \quad (13.2)$$

The fundamental postulate of the conditional probability interpretation is the formula for the probability  $P(B|C)$  that the answer to the question  $\hat{B}$  is yes if the answer to the question  $\hat{C}$  is yes:

$$P(B|C) = \langle \Psi | \hat{C} \hat{B} \hat{C} | \Psi \rangle / \langle \Psi | \hat{C} | \Psi \rangle. \quad (13.3)$$

The connection with the time problem is established by constructing a projection operator  $\hat{C}_T$ , corresponding to the question "Does an internal clock show the time T'?" (I chose the letter C for its two connotations: "the clock" and "the condition.")

The way in which the conditional probability interpretation works is best illustrated on simple models. The first model is the parametrized Newtonian particle moving in one dimension,

$$H := P_T + \frac{1}{2m} P_Q^2 + W(Q) = 0. \quad (13.4)$$

The second model is a system of two non-interacting particles, each of them with a single degree of freedom, kept in a stationary state:

$$\begin{aligned} H &:= h_T + h_Q - E \\ &:= \frac{1}{2M} P_T^2 + V(T) + \frac{1}{2m} P_Q^2 + W(Q) - E = 0. \end{aligned} \quad (13.5)$$

The position T of the first particle is cast in the role of the hand of a clock, and is thus called T.

The nature of the conditional probability interpretation is so clear from these examples that it is hardly necessary to spell out how the formalism looks in quantum gravity.

▪ **Parametrized Newtonian Particle.** This model is ideal for showing how the naive Schrödinger interpretation follows from the conditional probability interpretation. Let  $\mathcal{F}_T$  be the Hilbert space of square-integrable functions of T, and  $\mathcal{F}_Q$  the Hilbert space of square-integrable functions of Q. One constructs the projection operator

$$\hat{C}_T := |T'\rangle \langle T'| \quad (13.6)$$

in  $\mathcal{F}_T$  and the projection operator

$$\hat{B}_{Q'} := |Q'\rangle \langle Q'| \quad (13.7)$$

in  $\mathcal{F}_Q$ . When acting on  $\mathcal{F} = \mathcal{F}_T \otimes \mathcal{F}_Q$ ,  $\hat{C}_T$  acts as an identity operator on  $\mathcal{F}_Q$ , and



$\hat{B}_{Q'}$  as an identity operator on  $\mathcal{F}_T$ . Then  $\hat{C}_T$  is the questions "Is it T' o'clock?" and  $\hat{B}_{Q'}$  is the question "Is the particle at Q'?"

Let  $|\Psi\rangle$  satisfy the constraint (13.2), i.e., let  $\Psi(T, Q) := \langle T, Q | \Psi \rangle$  solve the Schrödinger equation. The conditional probability  $P(B_{Q'} | C_T)$  that the particle is at Q' at the time T' can be calculated from the fundamental formula (13.3). The answer

$$P(B_{Q'} | C_T) = |\Psi(T', Q')|^2 / \int_{\mathbb{R}} dQ |\Psi(T', Q)|^2 \quad (13.8)$$

is, up to the normalization factor, the same as that given by the naive Schrödinger interpretation, Eq. (12.2).

For the future discussion, note that the yes answer to the question  $\hat{C}_T$  determines a unique state in  $\mathcal{F}_T$  (i.e., the subspace  $\mathcal{F}_{T, C_T=1} \subset \mathcal{F}_T$  is one-dimensional), and similarly the yes answer to the question  $\hat{B}_{Q'}$  determines a unique state in  $\mathcal{F}_Q$ . (Actually, due to the continuous spectrum of the operators  $\hat{T}$  and  $\hat{Q}$ , these states are "improper" states which lie outside the respective Hilbert spaces  $\mathcal{F}_T$  and  $\mathcal{F}_Q$ .) However, when  $\hat{C}_T$  is considered as an operator in  $\mathcal{F} = \mathcal{F}_T \otimes \mathcal{F}_Q$ , the yes answer to the question  $\hat{C}_T$  means that the system lies in the infinitely-dimensional subspace  $\mathcal{F} = \mathcal{F}_T \otimes \mathcal{F}_Q$  of  $\mathcal{F}$  because nothing is yet known about the position of the particle. The operators  $\hat{C}_T$  and  $\hat{B}_{Q'}$  commute on  $\mathcal{F}$ .

▪ **Two Particles in a Stationary State.** One can apply the same strategy to the model (13.5). Let  $\mathcal{F}$  be the Hilbert space of square-integrable functions on the configuration plane (T, Q) and  $\mathcal{F}_0$  be the physical space of states satisfying the super-Hamiltonian constraint (13.2). The projection operators  $\hat{C}_T$  and  $\hat{B}_{Q'}$  are defined as before, Eqs. (13.6) - (13.7). They represent the questions "Is the first particle at T'?" and "Is the second particle at Q'?" The conditional probability  $P(B_{Q'} | C_T)$  that the second particle is at Q' if the first particle is at T' is again given by Eq. (13.8). This raises great expectations. The quadratic constraint (13.5) does not offer a privileged time like the Schrödinger constraint (13.4) does. The role of T and Q can easily be interchanged. Nevertheless, Eq. (13.8) seems to provide a probabilistic interpretation of the physical states  $|\Psi\rangle \in \mathcal{F}_0$ . It is not necessary to say beforehand what is time. It is not necessary for T to be a global time variable. (Indeed, for many potentials  $V(T)$  and  $W(Q)$ , as Hájíček's results indicate, it will not be a global time variable.) It is not necessary to cast the constraint (13.5) into a Schrödinger form before imposing it on the physical states. Equation (13.3) seems to be able to treat all conditions  $\hat{C}$  (all times) at once and on the same footing. All of this seems to suggest that the problems of time are pseudoproblems.

▪ **The Page-Wootters Treatment of the Model.** In my discussion of the system (13.5), I took the position of the first particle as a clock and the position of the second particle as a dynamical variable. Page and Wootters<sup>205</sup> generalize this procedure somewhat. To use the first particle as a clock, they choose a state  $|\phi_0\rangle \in \mathcal{F}_T$  and define the projector

$$\hat{C}_0 := |\phi_0\rangle \langle \phi_0| . \quad (13.9)$$

By evolving the state  $|\phi_0\rangle$  by the super-Hamiltonian  $\hat{H}$ , they arrive at a one-parameter family of projectors

$$\hat{C}_\tau := e^{-i\hat{H}\tau} \hat{C}_0 e^{i\hat{H}\tau} = e^{-i\hat{h}_T\tau} \hat{C}_0 e^{i\hat{h}_T\tau} . \quad (13.10)$$

They interpret  $\hat{C}_0$  as the question "Does the clock show the time 0?",  $\hat{C}_\tau$  as the question "Does the clock show the time  $\tau$ ?", and the parameter  $\tau$  as time.

Note that this interpretation is consistent only if  $\hat{C}_0$  is *not* a constant of the motion:

$$[\hat{C}_0, \hat{H}] = [\hat{C}_0, \hat{h}_T] \neq 0 . \quad (13.11)$$

If  $\hat{C}_0$  were a constant of the motion, Eq. (13.10) would yield  $\hat{C}_\tau = \hat{C}_0$ , i.e., the particle clock would not run.

The dynamical variable to be measured is a projector  $\hat{B} = B(\hat{Q}, \hat{P}_Q)$  constructed from the variables of the second particle. Take now a normalized state  $|\Psi\rangle \in \mathcal{F}_0$  and calculate the probability  $P(B|C_\tau)$  that the answer to  $\hat{B}$  is yes at the clock time  $\tau$ . Because  $\hat{B}$  commutes with  $\hat{C}_\tau$ ,  $\hat{B}$  commutes with  $\hat{h}_T$ ,  $\hat{C}_\tau$  commutes with  $\hat{h}_Q$ , and  $\hat{H} := \hat{h}_T + \hat{h}_Q - E$  annihilates the state  $|\Psi\rangle$ , one gets

$$P(B|C_\tau) = \langle \psi_0 | B_\tau | \psi_0 \rangle / \langle \psi_0 | \psi_0 \rangle , \quad (13.12)$$

where

$$|\psi_0\rangle := \langle \phi_0 | \Psi \rangle \in \mathcal{F}_Q \quad (13.13)$$

describes the state of the second particle and

$$\hat{B}_\tau := e^{i\hat{h}_Q\tau} \hat{B} e^{-i\hat{h}_Q\tau} \quad (13.14)$$

satisfies the Heisenberg equation of motion

$$\partial_\tau \hat{B}_\tau = \frac{1}{i} [\hat{B}_\tau, \hat{h}_Q] \quad (13.15)$$

in the parameter  $\tau$ . The conclusion drawn by Page and Wootters is that the conditional probability (13.12) depends on the time  $\tau$  as if the dynamical variable  $\hat{B}_\tau$  evolved according to Eq. (13.15) in spite of the fact that  $|\Psi\rangle$  is a *stationary state* (13.2) of the total system.

The conditional probability interpretation does not really save the naive Schrödinger interpretation. As far as it entails the naive Schrödinger interpretation, it shares its difficulties. As far as it reaches beyond it, by making temporal statements possible, it brings its own defects from hiding into sharp focus.

- **Inappropriate for a Klein-Gordon System.** When applied to a parametrized relativistic particle on a Minkowskian background, the conditional probability interpretation makes the prediction (13.8) for the particle to be found at the position  $Q'$  at the Minkowski time  $T'$ . This prediction is different from the accepted expression for the probability of the localization of a relativistic particle

- **Violation of the Constraints.** The formula (13.3), which the conditional probability interpretation takes as its fundamental postulate, is ordinarily derived from the rules governing measurements performed on a system by an outside apparatus. The ensemble  $\mathcal{E}$  of systems is prepared in a state  $|\Psi\rangle \in \mathcal{F}_0$ . First,  $\hat{C}$  is measured on  $\mathcal{E}$ . Those systems for which the answer to the question  $\hat{C}$  is yes are collected into a new ensemble  $\mathcal{E}_{C=1}$ . This ensemble is described by the state function

$$|\Psi_C\rangle = \hat{C}|\Psi\rangle \quad (13.16)$$

that is the projection of the initial state  $|\Psi\rangle$  into the subspace  $\mathcal{F}_{C=1} \subset \mathcal{F}$  spanned by the eigenfunctions of  $\hat{C}$  to the eigenvalue 1. (The selection of the subensemble  $\mathcal{E}_{C=1}$  of the original ensemble  $\mathcal{E}$  is sometimes called - appropriately or not - the reduction of the wave packet.) Second,  $\hat{B}$  is measured on  $\mathcal{E}_{C=1}$ . If the answer to the question  $\hat{B}$  is yes, the system falls within the ensemble  $\mathcal{E}_{B=1}$  described by the state function

$$|\Psi_B\rangle = \hat{B}|\Psi_C\rangle. \quad (13.17)$$

That is the projection of  $|\Psi_C\rangle$  into the subspace  $\mathcal{F}_{B=1} \subset \mathcal{F}$  spanned by the eigenfunctions of  $\hat{B}$  to the eigenvalue 1. The conditional probability (13.3),

defined as the ratio of the number of systems in  $\mathcal{E}_{B=1}$  and the number of systems in  $\mathcal{E}_{C=1}$ , is given by the norm of  $|\Psi_B\rangle$  divided by the norm of  $|\Psi_C\rangle$ .

From this description of the steps by which Eq. (13.3) is ordinarily derived it becomes clear that the conditional probability interpretation violates the constraints. The operators  $\hat{C}$  and  $\hat{B}$  act on  $\mathcal{S}$  and they are not required to commute with the constraint  $\hat{H}$ . Indeed, I pointed out that  $\hat{C}$  can play the role of a clock only if it does not commute with  $\hat{H}$ . Similarly, if the aim of the conditional probability interpretation is to describe how the Universe is changing, one must watch some  $\hat{B}$ 's which do not commute with  $\hat{H}$ , i.e., which are not constants of the motion.

However, if  $\hat{C}$  does not commute with  $\hat{H}$ , the measurement of  $\hat{C}$  will in general throw the state  $|\Psi\rangle$  out of the physical space,  $|\Psi_C\rangle \notin \mathcal{S}_0$ . Even if the state  $|\Psi_C\rangle$  were found to be in  $\mathcal{S}_0$ , the measurement of  $\hat{B}$  would in general throw it out of the physical space,  $|\Psi_B\rangle \notin \mathcal{S}_0$ . The conditional probability interpretation is thus based on a postulate whose derivation violates the constraints.

Page counters this objection by asserting that the intermediate state (13.16) is "actually only a calculational tool useful for writing the conditional probability in the simple form" and that "the 'collapse' of the wavefunction ... should be merely thought of as part of one's computational process, not as a physical process, so it should not be surprising that the result is rather 'unphysical'."<sup>206</sup>

I fear that such an attempt to brush aside the violation of the constraint as a mere verbal artifice does not stand closer scrutiny. I shall try to show that this mechanism of violating the constraints is actually responsible for the fact that the conditional probability interpretation predicts *physically wrong* propagators for the simplest model system, namely, the parametrized Newtonian particle.

- **Wrong Propagators.** Curiously enough, the conditional probability interpretation was used by its proponents only to calculate the conditional probabilities of  $\hat{B}$  at a *single* instant  $\hat{C}_T$ , but never applied to answering the fundamental *dynamical* question of the internal Schrödinger interpretation, namely, "If one finds the particle at Q' at the time T', what is the probability of finding it at Q" at the time T" > T'?"

Let me pose this question for a parametrized Newtonian particle within the framework of the conditional probability interpretation. Let  $\hat{C}_T$  and  $\hat{B}_Q$  be the projectors defined by Eqs. (13.6) - (13.7). Then their product

$$\hat{A}_{T'Q'} := \hat{B}_Q \hat{C}_T \tag{13.18}$$

is the projector expressing the question "Is the particle at Q' at the time T'?" Because the projectors  $\hat{B}_Q$  and  $\hat{C}_T$  commute on  $\mathcal{S}$ , the definition (13.18) makes sense. Note that while the projectors  $\hat{B}_Q$  and  $\hat{C}_T$ , taken separately generate

multidimensional subspaces  $\mathcal{S}_{B_Q,=1}$  and  $\mathcal{S}_{C_T,=1}$  of  $\mathcal{S}$ , their product (13.18) generates a ray:  $\mathcal{S}_{A_{T',Q',=1}}$  is a one-dimensional subspace of  $\mathcal{S}$ . (Of course, strictly speaking, none of the spaces  $\mathcal{S}_{B_Q,=1}$ ,  $\mathcal{S}_{C_T,=1}$ , and  $\mathcal{S}_{A_{T',Q',=1}}$  is a subspace of  $\mathcal{S}$ , because the operators  $\hat{T}$  and  $\hat{Q}$  on  $\mathcal{S}$  have a continuous spectrum.)

The probability sought by the fundamental dynamical question is given in the conditional probability interpretation framework by the formula

$$P(A_{T''Q''}|A_{T'Q'}) = \langle \Psi | \hat{A}_{T'Q'} \hat{A}_{T''Q''} \hat{A}_{T'Q'} | \Psi \rangle / \langle \Psi | \hat{A}_{T'Q'} | \Psi \rangle . \quad (13.19)$$

The physical answer to the fundamental dynamical question should, of course, be the square of the non-relativistic propagator. Instead, Eq. (13.19) yields,

$$P(A_{T''Q''}|A_{T'Q'}) = |\delta(T'-T'')\delta(Q'-Q'')|^2 . \quad (13.20)$$

In brief, the conditional probability interpretation prohibits the absolute time to flow and the Newtonian particle to move!

The application of the fundamental formula to the Page-Wootters treatment of the two-particle model does not fare any better. For simplicity, take the initial state of the clock particle to be  $|\phi_0\rangle = |T'\rangle$  and ask again about the position  $Q$  of the second particle. The projector

$$\hat{A}_{T'Q'} := \hat{B}_Q \hat{C}_{\tau'} , \quad (13.21)$$

with  $\hat{C}_{\tau'}$  given by Eq. (13.10), articulates the question "Is the second particle at the position  $Q'$  at the clock time  $\tau'$ ?" The probability  $P(A_{T''Q''}|A_{T'Q'})$  predicted by the conditional probability interpretation is

$$P(A_{T''Q''}|A_{T'Q'}) = |\langle T' | e^{-i\hat{h}_T(\tau''-\tau')} | T' \rangle|^2 |\delta(Q'-Q'')|^2 . \quad (13.22)$$

Again, Eq. (13.22) prohibits the second particle to move. For  $\hat{h}_T = \hat{P}_T$ , Eq. (13.22) gives the old result (13.20).

One may wonder why Eq. (13.8) for the Newtonian particle seemed to replicate the standard quantum dynamics of  $Q'$  with  $T'$ , while Eq. (13.20) miserably failed to do the same. The answer is that the measurements of  $\hat{B}_Q$  in Eq. (13.8) are done on an ensemble  $\mathcal{E}_{C_T,=1}$  of particles whose time is known but the position  $Q'$  is as yet undetermined, while the measurements of  $\hat{A}_{T''Q''}$  in Eq. (13.20) are done on an ensemble  $\mathcal{E}_{A_{T',Q',=1}}$  of particles that were already found to be at  $Q'$  at the time  $T'$ . The measurement of  $\hat{A}_{T'Q'}$  with the result 1 is a complete

measurement which uniquely fixes the state  $|T',Q'\rangle$  of the ensemble  $\mathcal{E}_{A_{T'Q'-1}}$  reducing thereby the original physical state  $|\Psi\rangle \in \mathcal{F}_0$  to an unphysical state  $|T',Q'\rangle \notin \mathcal{F}_0$  that has nothing to do with the original  $|\Psi\rangle$ . The unphysical result (13.20) is thus a direct consequence of the violation of the constraint by the first measurement, the violation which Page tried to dispute away as a purely computational device.

I regard Eq. (13.20) as a reduction ad absurdum of the conditional probability interpretation. If the interpretation does not yield an expected physical prediction for a parametrized Newtonian particle, how can one rely on it in quantum gravity? Of course, one can try to modify the conditional probability interpretation, say, by projecting the state back into the physical space  $\mathcal{F}_0$  each time the measurement of a projector  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$ , ... brings it out of the physical space. I better abstain from analyzing the shortcomings of such a scheme before someone seriously proposes it.

#### 14. Sum-Over-Histories Interpretation

In canonical quantization of a Newtonian system, path integrals are tools for calculating the propagator which determines the change of the state in the Hilbert space from one instant of time to another. The idea pursued by the sum-over-histories interpretations is that path integrals somehow make sense even outside the Hilbert space framework and provide an interpretation for quantum systems which do not automatically carry with them the notions of time and of the Hilbert space.

Such an approach to quantum gravity was advocated by Teitelboim who based on it a statistical interpretation of the scattering process (in proper time) in which the Universe starts in a velocity-dominated singularity and, encountering the scalar potential of the Wheeler-DeWitt equation, ends in a velocity-dominated singularity.<sup>209-211</sup> A more elaborate sum-over-histories interpretation of quantum gravity was offered by Hartle. Hartle's interpretation is a part of an ambitious program aiming at connecting quantum mechanics with the theory of initial conditions of the Universe.<sup>75,130,131,169,212-217</sup> I shall restrict my discussion strictly to those aspects of the program which deal with the problem of time.

At the technical level, Hartle used path integrals for reinterpreting ordinary nonrelativistic quantum mechanics,<sup>212</sup> and then applied the same scheme to the toy universe of two particles kept in a stationary state.<sup>75</sup> At a more heuristic level he argued that the main ideas of his interpretation can be generalized from the toy model to quantum gravity.<sup>130,213,214,216</sup> I shall summarize the essential features of Hartle's interpretation.

▪ **Sum-Over-Histories Interpretation for Two Model Systems.** The model systems studied by Hartle are the parametrized Newtonian particle (13.4), and two noninteracting particles, with coordinates T and Q, kept in a stationary state (13.5). The second model is thus the same as that used by Page and Wootters, and it is a specialization of the model (3.9) considered in the semiclassical interpretation.

The fundamental rules of the sum-over-histories interpretation are the following:

I) *Observables.* The basic observables are projectors associated with regions of the configuration space (T,Q). A detector measuring an observable registers if the system crosses the region at least once but possibly many times, and it does not register if the system avoids the region.

II) *Probability Amplitudes.* These are to be calculated by Feynman's standard prescriptions in which the super-Hamiltonian is treated as an ordinary Hamiltonian and the state is propagated in a label time  $\tau$ . (For the super-Hamiltonian (13.4), the label time flows at the same pace as the internal Newtonian time T; for the model (13.5), it corresponds to the Newtonian time of an outside observer.)

III) The *label time* should be treated as an *unobservable label*. This means that the amplitudes must be summed over  $\tau$  *before* being squared to yield joint probabilities. Both positive and negative  $\tau$ 's are allowed; because the label time order is unobservable, the amplitudes corresponding to the positive and negative  $\tau$  must also be summed over before the probabilities are determined.

To extract from these rules a probabilistic interpretation, one *prepares a state*  $|\Phi\rangle$  by imposing *conditions* which the *histories* (the paths of the system in the configuration space (T,Q)) must satisfy. The conditions assert that the histories must pass through given regions; the preparation of the state means that the detectors associated with those regions must register. After that, one fixes an "*exhaustive and exclusive*" set of observations consistent with the conditions. This can be done by removing from the configuration space the regions which defined the conditions, and breaking the remainder into a collection of non-overlapping regions. For definiteness, let there be only three such regions, 1, 2, and 3. The exhaustive and exclusive set of observations corresponds then to the following possible combinations of the responses of the three detectors:

- (0): None of the three detectors registers.
- (1): One and only one of the detectors registers.

- (1;1): This is the detector 1.
- (1;2): This is the detector 2.
- (1;3): This is the detector 3.
- (2): Two and only two of the detectors register.
  - (2;12): These are the detectors 1 and 2.
  - (2;13): These are the detectors 1 and 3.
  - (2;23): These are the detectors 2 and 3.
- (3): All three detectors register.

By performing these exhaustive and exclusive observations on an ensemble of systems each of which is prepared to satisfy the same conditions, one can experimentally determine the relative frequencies of the possible outcomes. The sum-over-histories interpretation predicts what these relative frequencies ought to be by following the rules I to III. The results are obtained by evaluating path integrals, but they can be summarized in the language of operators on an auxiliary Hilbert space  $\mathcal{F}$  of square integrable functions of T and Q. I shall write down two such probabilities, one from the class (1),<sup>75</sup>

$$\begin{aligned} P(1;3|\Phi) &= \mathcal{N} \int_3 dT dQ |\langle TQ | \hat{K}_+ | \Phi \rangle + \langle TQ | \hat{K}_- | \Phi \rangle|^2 \\ &= \mathcal{N} \int_3 dT dQ |\langle TQ | \hat{K} | \Phi \rangle|^2 \end{aligned} \quad (14.1)$$

and another one from the class (2),<sup>75</sup>

$$P(2;31|\Phi) = \mathcal{N} \int_1 dT dQ |\langle TQ | \hat{K}_+ \hat{\Pi}_3 \hat{K}_+ | \Phi \rangle + \langle TQ | \hat{K}_- \hat{\Pi}_3 \hat{K}_- | \Phi \rangle|^2 + (1 \leftrightarrow 3). \quad (14.2)$$

The common normalization factor  $\mathcal{N}$  is fixed by the requirement that the probabilities of all the listed exhaustive and exclusive options must sum to 1. The operators  $\hat{K}_\pm$  are the forward and backward propagators

$$\hat{K}_\pm = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \theta(\pm\tau) e^{-i\hat{H}\tau} \quad (14.3)$$

obtained by integrating the familiar forward and backward Schrödinger propagators over the positive (negative) label time  $\tau$ , as indicated by the step function  $\theta$ . The operator  $\hat{\Pi}_3$  is the projector



$$\hat{\Pi}_3 := \int_3 dT dQ |TQ\rangle \langle TQ| \quad (14.4)$$

into the region 3. The state vector  $|\Phi\rangle$  describes an ensemble of systems prepared in the state satisfying the given conditions.

The construction of the probabilities (14.1) and (14.2) allows an intuitive interpretation. The probability  $P(1;3|\Phi)$  that in the state  $|\Phi\rangle$  one and only one detector registers, and that this is the detector 3, is obtained by squaring the sum of two probability amplitudes, the first one for the state  $|\Phi\rangle$  to be propagated forward to a point T,Q of the region 3, the second one for the state to be propagated backward. The result is the same as if the state were propagated by

$$\hat{K} := \hat{K}_+ + \hat{K}_- = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i\hat{H}\tau} = \delta(\hat{H}). \quad (14.5)$$

The probability  $P(2;31|\Phi)$  that in the state  $|\Phi\rangle$  two and only two detectors register, and that these are the detectors 1 and 3, is obtained by summing up two *probabilities*. Each of them is the square of the sum of two *probability amplitudes*. The first probability amplitude is for the state  $|\Phi\rangle$  to propagate forward to the region 3, to project to that region, and the projected state to propagate forward to a point of the region 1. The second probability amplitude corresponds to the same trajectory being traversed backwards. These two probability amplitudes yield the first probability. The second probability is constructed in the same way, but the regions 1 and 3 are now interchanged, the region 1 becoming an intermediate region, and the trajectory ending at a point of the region 3. The integrations over  $\tau$  which yield the forward and backward propagators, and the sum over these two kinds of propagations, both the integration and the sum taking place in the probability *amplitudes*, are the manifestation of the rule III) about the unobservability of the label time.

A major part of Hartle's analysis is devoted to the recovery of the Schrödinger-Heisenberg quantum dynamics from the sum-over-histories interpretation. For this purpose, a system of thin-slab detectors is arrayed to cover a hypersurface in the (T,Q) plane. By analyzing the joint response of such detectors, Hartle draws a number of conclusions:

- In general, there is no Hilbert space formulation associated with the hypersurface. This is because the histories do not in general intersect the hypersurface once and only once.

- Consider only the hypersurfaces  $T = \text{const}$ . For a parametrized

Newtonian particle, there are states for which the clock variable  $T$  is forever precisely correlated with the label time  $\tau$ . In such a state, the response of the thin-slab detectors can be exactly described in terms of a state evolving in accordance with the Schrödinger equation. For the model with two particles in a stationary state, there are states ("good clock states") in which  $T$  and  $\tau$  are correlated for a physically interesting interval of time. The response of the detectors is then at least approximately obtained from a Schrödinger equation.

The sum-over-histories interpretation is clearly preferable to the semiclassical interpretation because *every* solution of the constraint equation, not only a "good clock state" solution, has a probabilistic interpretation. It is also preferable to the naive Schrödinger interpretation because it is able to describe correlations among responses of two or more detectors. However, it does not, in my opinion, escape the issues of time:

- **Multiple Choice Problem.** The predictions of the sum-over-histories interpretation depend on which one of the classically equivalent super-Hamiltonians is chosen in the canonical path integral. The interpretation thus requires a commitment to one definite form of the constraints. The presumption seems to be that in canonical gravity one should prefer the standard quadratic form (1.2) of the super-Hamiltonian, i.e., work within the Wheeler-DeWitt framework. I take this to be a satisfactory answer to the multiple choice problem, as long as one sticks to it all the way through and does not switch the super-Hamiltonian somewhere in the middle of the argument (e.g., use the extrinsic time super-Hamiltonian when constructing a "good clock state").

- **The Functional Evolution Problem.** In Hartle's models, the label time  $\tau$  is a single parameter. In geometrodynamics,  $\tau$  is replaced by an external many-fingered time: each choice of  $\tau$  corresponds to a choice of the lapse and shift multipliers, i.e., to a choice of the foliation. When evaluating the path integral, one must make a definite choice. The resulting integral can be expected to depend on the choice of the foliation. (The formal arguments that path integrals are "gauge independent" are hardly reliable at this level of discussion.) This is how the functional evolution problem reappears in the sum-over-histories framework.

- **Violation of the Constraints.** Hartle repeatedly emphasizes that there is no Hilbert space associated with a hypersurface in the  $(T, Q)$  plane. There is, however, a Hilbert space behind the formalism, namely, the Hilbert space  $\mathcal{F}$  of square-integrable functions over the whole  $(T, Q)$  plane. Note that Eqs. (14.1) - (14.5) which determine the detector responses involve operations in  $\mathcal{F}$ . One

should pay close attention to the role of  $\mathcal{F}$  in the sum-over-histories interpretation.

When discussing the naive Schrödinger interpretation and its conditional probabilities refinement, I pointed out that the Hilbert space of square-integrable functions over the entire configuration space(time) does not seem to be a natural arena for interpreting a constrained system. The sum-over-histories interpretation, by sharing this Hilbert space with the two previous interpretations, shares with them also the difficulties. To find what they are, I analyze Eqs. (14.1) - (14.5) from the standpoint of canonical quantum mechanics.

When only one of the detectors registers, the probability that it is a particular detector (say, the detector 3) replicates the probability postulated by the naive Schrödinger interpretation. Note that the state  $|\phi\rangle \in \mathcal{F}$  prepared by the conditions of the experiment does not need to satisfy the super-Hamiltonian constraint. However, the state  $|\Psi\rangle = \hat{K}|\phi\rangle$  which enters into Eq. (14.1) does satisfy the constraint,

$$\hat{H}|\Psi\rangle = \hat{H}\hat{K}|\phi\rangle = \hat{H}\delta(\hat{H})|\phi\rangle = 0, \quad (14.6)$$

because the forward and backward propagators  $\hat{K}_+$  and  $\hat{K}_-$  combine into the projector  $\hat{K} = \delta(\hat{H})$  into the physical space  $\mathcal{S}_0$ . Equation (14.1) thus directly corresponds to Eq. (12.2).

The naive Schrödinger interpretation does not say what happens to the state  $|\Psi\rangle$  when the system is found in the region 3, and what would then be the probability of another observation. It is to the credit of the sum-over-histories interpretation that it does tell one the probability with which another detector registers. However, its answer, Eq. (14.2), does not correspond to the idea that the state of the system satisfies the super-Hamiltonian constraint between observations. This is brought in by the fact that the forward and backward propagators no longer neatly combine into the projector  $\hat{K}$ . The constituents of Eq. (14.2) no longer satisfy the constraint:  $|\phi\rangle$  was not prepared to do so, and  $\hat{K}_+$  does not project it into the physical space; even if it did,  $\hat{\Pi}_3$  would kick the vector  $\hat{K}_+|\phi\rangle$  again off the physical space, and the subsequent propagator  $\hat{K}_+$  would do nothing to bring it back. The same remarks apply to the remaining three terms in Eq. (14.2). It is thus profoundly puzzling what connection the sum-over-histories interpretation has with the space  $\mathcal{S}_0$  of states which satisfy the super-Hamiltonian constraint. Hartle devotes considerable attention to setting up the path integral formulation so that the constraints are satisfied for the state function that determines the response of a single detector. On the other hand, he evidently regards as unimportant whether the constraints are satisfied by the intermediate states connecting two or more detectors. From the point of view of the canonical formalism, the sum-over-histories interpretation has essentially the

same defect as the conditional probability interpretation: the structure of Eq. (14.2) indicates that the observations are not performed so that the state is always kept in the physical space  $\mathcal{F}_0$ .

This conclusion had to be expected from the beginning. The detectors designed to measure "Q and T" must be at odds with the constraints. Effectively, the constraints prohibit an independent determination of Q and T; they allow only for the measurements of "Q at T". This strongly suggests that the probabilistic interpretation of a constrained system should be based on *fluxes through hypersurfaces* rather than on *densities in space*.

- **Why Should one Believe in Probability Amplitudes?** For a relativistic particle on a stationary background, the Hilbert space  $\mathcal{F}_0^+$  of positive-energy one-particle states can be constructed from the wave function  $\Psi(T, Q)$  and its normal rate of change (restricted by the positive-energy requirement) on an arbitrary spacelike hypersurface. However, the inner product of two wave functions on a given hypersurface is given by the Klein-Gordon rule (9.5) rather than by the "square of the amplitude" rule postulated by the sum-over-histories interpretation. The "square-of-the-amplitude" rule is appropriate only if the state  $\Psi$  is given in the Schrödinger position representation and if it is propagated according to the Schrödinger equation. Neither of these conditions is satisfied for the wave function  $\Psi(T, Q)$ . As a result, the rules I) - III) do not lead to the one-particle propagator in the Hilbert space  $\mathcal{F}_0^+$  of positive-energy solutions of the Klein-Gordon constraint.

This example illustrates the point that there is no reason to believe that the Feynman rule which treats the state function as the probability amplitude provides an appropriate framework for quantizing a generic constrained system. In particular, why should it work in quantum gravity?

## 15. Frozen Time Formalism and Evolving Constants of the Motion

In a gauge theory, only those dynamical variables that are left unchanged by gauge transformations have a physical meaning. Such variables are called *observables*. Thus, the electric and magnetic field strengths in electrodynamics are observables, while the scalar and vector potentials are not.

In canonical formalism, gauge transformations are generated by constraints that are linear in the momenta. These constraints move a point in the phase space along an orbit of the gauge group. Two points on the same orbit are physically indistinguishable; they represent merely two equivalent descriptions of the same physical state. An observable should not depend on what description of the state one chooses. This means that it must be the same along the orbit, i.e., that its Poisson bracket with the constraints must vanish.

General relativity is a constraint theory par excellence: All its physical content is encoded in the constraints. Previous experience with ordinary gauge theories led to the idea that the observables  $F$  in general relativity are those dynamical variables that have vanishing Poisson brackets with all the constraints:

$$\{F, H_a(x)\} = 0 = \{F, H(x)\} . \quad (15.1)$$

It is generally agreed that the supermomentum constraint (1.1) generates a gauge, namely, the group of diffeomorphisms  $\text{Diff}\Sigma$  of spatial coordinates, and that any observable in general relativity must be invariant under  $\text{Diff}\Sigma$ . However, the super-Hamiltonian constraint plays a different role: It generates the dynamical change of the geometrodynamical variables from one hypersurface to another. Any dynamical variable which commutes with the super-Hamiltonian must be the same on every hypersurface  $\Sigma \rightarrow \mathcal{M}$ , i.e., it must be a *constant of the motion*. To maintain that the only observable quantities are those that commute with all the constraints seems to imply that the Universe can never change. For this reason, this standpoint on observables was dubbed the *frozen time formalism*.<sup>218-223</sup> The frozen time formalism never successfully explained the evolution that we see all around us.

A similar difficulty appears already in simple parametrized dynamical systems, as the Newtonian particle. An observable defined as a dynamical variable that has the vanishing Poisson bracket with the super-Hamiltonian constraint (1.11) must necessarily be a constant of the motion. Nevertheless, in ordinary Newtonian dynamics one happily works with dynamical variables that change with time, like the position of the particles, and assumes that such variables can be measured. This indicates that the transformations generated by the super-Hamiltonian should not be interpreted as gauge transformations: Two points on the same orbit of the super-Hamiltonian are two events in the dynamical evolution of the system. Such events are physically distinguishable rather than being two descriptions of the same physical state. For this reason, I took the position in the rest of this paper that observables are to be defined as those dynamical variables that have vanishing Poisson brackets with the gauge constraints, but not necessarily with the super-Hamiltonian constraint (see Section 6, Eq. (6.3)).

Recently, Rovelli proposed a reinterpretation of the frozen time formalism that allows one to describe within it the dynamical change of a system.<sup>224-226</sup> He introduced the scheme on a model of two harmonic oscillators in a stationary state<sup>225</sup> and then attempted to generalize it to an arbitrary finite-dimensional system whose dynamics is generated by a single super-Hamiltonian constraint.<sup>226</sup> His main objective was to show that one does not need a global time function for interpreting quantum dynamics. Dynamics is introduced by the concept of an

*evolving constant of the motion.*

▪ **Evolving Constants of the Motion.** I shall explain how the scheme is expected to work for a system with the super-Hamiltonian  $H(Q,P)$  that admits a global time function  $T(Q,P)$ . This assumption means that each dynamical trajectory intersects every hypersurface

$$T(Q,P) = t = \text{const} \quad (15.2)$$

once and only once, i.e., that

$$\{T, H\} \neq 0 . \quad (15.3)$$

(Throughout his work, Rovelli does not distinguish the weak equations from the strong ones, and I shall not try to fix this deficiency.)

Let  $F(Q,P)$  be an arbitrary dynamical variable that is not necessarily a constant of the motion:

$$\{F, H\} \neq 0 . \quad (15.4)$$

Choose a fixed value of  $t$  and define a phase-space function  $F_t(Q,P)$  by the requirements that (I)  $F_t(Q,P)$  coincide with  $F(Q,P)$  when restricted to the hypersurface (15.2), and that (II)  $F_t(Q,P)$  be constant along each classical trajectory:

$$\{F_t, H\} = 0 . \quad (15.5)$$

If  $T$  is a global time function, Eq. (15.3), these two requirements determine  $F_t(Q,P)$  uniquely. When repeated for each value of  $t$ , the procedure replaces a dynamical variable  $F(Q,P)$  by a one-parameter family  $F_t(Q,P)$ ,  $t \in (-\infty, \infty)$  of observables. The first requirement amounts then to the statement that

$$F_{t=T(Q,P)} = F(Q,P) . \quad (15.6)$$

Dynamics does not tell one how a single  $F_t(Q,P)$  changes when evolved by  $H$ , because each  $F_t(Q,P)$  is a constant of the motion and hence does not change at all, but it tells one how the observables  $F_t(Q,P)$  depend on the parameter  $t$ . By taking the Poisson bracket of Eq. (15.6) with the super-Hamiltonian, one learns that

$$\left. \frac{dF_t(Q,P)}{dt} \right|_{t=T(Q,P)} = \{T, H\}^{-1} \{F, H\} . \quad (15.7)$$

This means that the observables  $F_t(Q,P)$  change with  $t$  in the same rate at which the dynamical variable  $F$  evolves in the internal time  $T$ .

Rovelli calls the one-parameter family of observables  $F_t(Q,P)$  an *evolving constant of the motion*. For a classical system with a global time function, the evolution of dynamical variables can be described in terms of evolving constants of the motion.

▪ **Quantization.** Rovelli suggests that the algorithm used to define the evolving constants of the motion can be repeated at the quantum level. One simply turns the classical variables  $H(Q,P)$ ,  $T(Q,P)$  and  $F(Q,P)$  into operators  $H(\hat{Q},\hat{P})$ ,  $T(\hat{Q},\hat{P})$  and  $F(\hat{Q},\hat{P})$  and defines the *quantum evolving constant of the motion* by the requirements

$$[\hat{F}_t, \hat{H}] = 0 \quad (15.8)$$

and

$$F_{t=T(\hat{Q},\hat{P})}(\hat{Q},\hat{P}) = F(\hat{Q},\hat{P}) \quad (15.9)$$

that mimic the classical equations (15.5) - (15.6). The change of the quantum evolving constant of the motion  $F_t(\hat{Q},\hat{P})$  with  $t$  describes the quantum dynamics of the system.

Rovelli conjectures that this procedure enables one to solve the Hilbert space problem. For a generic super-Hamiltonian  $\hat{H}$ , the physical space  $\mathcal{S}_0$  of state functions  $\Psi(Q)$  that satisfy the quantum constraint

$$\hat{H}\Psi(Q) = 0 \quad (15.10)$$

does not have any obvious Hilbert space structure. Rovelli proposes to choose a "complete set" of dynamical variables  $F_k(\hat{Q},\hat{P})$  and turn all of them into corresponding quantum constants of the motion  $F_{k,t}(\hat{Q},\hat{P})$ . His conjecture is that there exists a unique inner product in  $\mathcal{S}_0$  such that all  $F_{k,t}(\hat{Q},\hat{P})$  operators are self-adjoint under this inner product.

I want to show that Rovelli's procedure is ambiguous and that, even if its ambiguities were somehow resolved, it is not likely to solve any one of the outstanding problems of time.

- **Quantum Evolving Constants of the Motion are Ambiguous.**

Rovelli's procedure starts by turning the classical variables  $H(Q,P)$ ,  $T(Q,P)$  and  $F(Q,P)$  into operators. Without some further guiding principle the factor ordering of the operators  $\hat{Q}$  and  $\hat{P}$  is arbitrary and so are the resulting operators  $\hat{H}$ ,  $\hat{T}$  and  $\hat{F}$ . Even if these factor ordering problems were resolved, there is an additional problem of writing down Eq. (15.9) (Hájíček<sup>227</sup>). The time operator  $\hat{T} = T(\hat{Q}, \hat{P})$  does not commute with  $\hat{Q}$  and  $\hat{P}$ , and the replacement of the classical parameter by the operator  $\hat{T}$  is thus ambiguous. Even if this problem were resolved, the commutator condition (15.8), unlike its classical counterpart (15.5), is not sufficient to determine the operator  $\hat{F}_t$  uniquely. Equation (15.8) determines the quantum evolving constant of the motion only if the super-Hamiltonian has the Schrödinger form (1.11) with respect to the time function  $T$ .

In view of all these factor-ordering problems, it seems best to drop out the requirement (15.9) and to formulate the quantum problem somewhat differently. One can first, at the classical level, construct the classical evolving constant of the motion  $F_t(Q,P)$  which satisfies the classical requirements (15.5) - (15.6). One can then pose the problem of factor ordering this one-parameter family of classical observables so that they remain observables even in the quantum theory. Unfortunately, one lacks an algorithmic procedure for achieving this aim. This is a manifestation of the problem of time: In the Schrödinger case, the algorithm depends on the identification of the true Hamiltonian as the variable canonically conjugate to the time. The evolving constants of the motion are then constructed by means of the standard evolution operator that requires one to decide on a factor ordering of a single quantity, namely, the true Hamiltonian of the system.

- **The Hilbert Space Problem.** I want to illustrate why Rovelli's conjecture on resolving the Hilbert space problem is likely to fail. To do that, I choose a simple model with a global time function, namely, a relativistic particle moving on a dynamical background.

I select an intrinsic time function  $T(Q)$  and label the worldlines of the reference frame that is orthogonal to the  $T$ -foliation by the comoving coordinates  $q^a(Q)$ , as in Eq. (6.22). Let  $p_a(Q,P)$  be the momentum canonically conjugate to  $q^a$ . The dynamical variables  $q^a(Q)$ ,  $p_a(Q,P)$  form a complete set of fundamental canonical variables. Their Poisson brackets are

$$\{q^a, p_b\} = \delta_b^a, \quad \text{etc.} \quad (15.11)$$

The variables  $q^a$ ,  $p_a$  are not constants of the motion, but following Rovelli's algorithm one can replace each of them by the corresponding evolving constant of the motion, i.e., by the one-parameter family of constants of the motion  $q_t^a$ ,  $p_{t,a}$ . For each value of  $t$ , these constants of the motion have the same Poisson brackets as the fundamental canonical variables that they represent:



$$\{q_t^a, p_{t b}\} = \delta_b^a, \quad \text{etc.} \quad (15.12)$$

One knows from the vanHove theorem<sup>228</sup> that one cannot preserve the whole classical Poisson algebra upon quantization. A crucial decision which one faces is the selection of a suitable subalgebra that is taken into quantum theory without modification. Though Rovelli does not explicitly state this, I presume (as it is common in quantum mechanics of particle systems), that he wants to take over into quantum theory the algebra (15.12) of the complete set of observables. If so, he must factor order the variables  $q_t^a(\hat{Q})$  and  $p_{t a}(\hat{Q}, \hat{P})$  so that they both preserve their classical algebra,

$$\frac{1}{i} [q_t^a(\hat{Q}), p_{t b}(\hat{Q}, \hat{P})] = \delta_b^a, \quad \text{etc.}, \quad (15.13)$$

and remain the constants of the motion,

$$[q_t^a(\hat{Q}), H(\hat{Q}, \hat{P})] = 0 = [p_{t a}(\hat{Q}, \hat{P}), H(\hat{Q}, \hat{P})]. \quad (15.14)$$

This is a non-trivial task because  $q_t^a(Q)$ , and  $p_{t a}(Q, P)$  are complicated functions of the classical arguments  $Q^A$  and  $P_A$  (thus, e.g.,  $p_{t a}(Q, P)$ , unlike  $p_a(Q, P)$ , is no longer linear in the momenta  $P_A$ ). I already pointed out that Rovelli's algorithm for achieving (15.14) does not in general work. This is not, however, my present concern: *Assume* that, by hook or by crook, one finds a factor ordering such that Eqs. (15.13) - (15.14) hold for the quantum operators.

Rovelli's conjecture is that one can endow  $\mathcal{F}_0$  with an inner product such that  $\hat{q}_t^a$  and  $\hat{p}_{t a}$  are self-adjoint operators. Because the Klein-Gordon operator  $\hat{H}$  is a second-order differential operator, the solution space  $\mathcal{F}_0$  is quite big: it contains solutions corresponding to both the positive and negative energies. It is quite possible that there does not exist an inner product in the whole of  $\mathcal{F}_0$  under which the operators  $q_t^a$  and  $p_{t a}$  are self-adjoint. The crux of the Hilbert space problem, however, lies elsewhere:

Choose another time function,  $\bar{T}(Q)$ , and repeat the whole procedure. Get thereby another set,  $\hat{\bar{q}}_t^a$  and  $\hat{\bar{p}}_{t a}$ , of quantum observables. Keep repeating this for all possible intrinsic time functions  $T(Q)$ . Now, against Rovelli's conjecture, let me pose a counterconjecture: It is impossible to turn all the operators  $\hat{q}_t^a$ ,  $\hat{p}_{t a}$  corresponding to all possible intrinsic time functions  $T(Q)$  into self-adjoint operators by a *single* choice of an inner product in  $\mathcal{F}_0$ .

If it were possible, one would have a consistent one-particle interpretation of the Klein-Gordon theory on a dynamical background, the interpretation that would be independent of the chosen foliation. Nobody has so far succeeded in constructing such an interpretation. This is the challenge which the relativistic

particle puts to Rovelli's proposal. The problem that I just spelled out is a modified version of the multiple choice problem for the relativistic particle discussed in Section 6.

- **Multiple Choice Problem.** Before going any further, let me note that Rovelli's formalism has yet another multiple choice problem: It depends on which one of the classically equivalent super-Hamiltonians one turns into an operator  $\hat{H}$  that defines the quantum theory. I presume that in quantum geometrodynamics most people would be inclined to solve this problem by insisting that the super-Hamiltonian constraint be imposed in its familiar quadratic form.

- **Global Time Problem.** When there is no global time function, the replacement (15.5) - (15.6) of a dynamical variable  $F$  by an evolving constant of the motion  $F_t$  breaks down. Nevertheless, Rovelli claims that one can give a dynamical interpretation of such systems without a global time function. He bases his claim on a discussion of a single highly special model, namely, a two-oscillator system. Hájiček<sup>227</sup> pointed out an error in Rovelli's construction of an allegedly self-adjoint observable based on a nonglobal time function in this model. In response to Hájiček's criticism, Rovelli modified the factor ordering of his observable.<sup>229</sup> In my opinion, the physical interpretation of Rovelli's observable remains problematic. Moreover, the two-oscillator model has such a high degree of symmetry that drawing any general conclusions from its treatment is clearly premature. Hartle<sup>230</sup> raised the question whether Rovelli's interpretation based on a non-global time function is consistent even for a parametrized Newtonian particle.

- **Functional Evolution Problem.** Rovelli never got so far as to discuss the evolving constants of the motion for a field-theoretical system with a many-fingered time. Hartle pointed out that the generalization of Rovelli's procedure to such systems may present difficulties.<sup>230</sup>

Rovelli's procedure enables one to discuss classical dynamics of systems with a global time function in terms of constants of the motion. However, it does not seem to be able to solve a single one of the outstanding problems of time. Indeed, Rovelli himself did not develop his proposal to the point at which it becomes clear where the problems of time lie.

## 16. Summary

In general relativity, dynamics is entirely generated by constraints. The

dynamical data do not explicitly include a time variable. I identified several major problems which this situation causes in quantum gravity. A classical obstacle to interpreting the quantum formalism is the possibility that there is *no global time function* on the phase space of the theory, and the associated *sandwich problem*. Even if a global time function exists, quantum theory may depend on its particular choice; this constitutes the *multiple choice problem*. Moreover, even when one sticks to a definite choice of the many-fingered time, one still faces the *problem of functional evolution*: the propagation of the state functional may depend on the foliation connecting the initial hypersurface with the final hypersurface. Those interpretations of quantum gravity that do not try to separate time from dynamical variables trade the multiple choice problem for the *Hilbert space problem*: how does one endow the space of solutions of the quantum constraints with a Hilbert space structure?

I classified ten major attempts at interpreting quantum gravity into three classes: I) *An internal time framework*, II) *the Wheeler-DeWitt framework*, and III) *quantum gravity without time*.

The first class of interpretations tries to identify time and bring the super-Hamiltonian constraint into the Schrödinger form prior to quantization. I reviewed various attempts to construct time from the geometric variables in the *internal Schrödinger interpretation* (the intrinsic and extrinsic times of minisuperspace and midisuperspace models, of linearized gravity, and the mean extrinsic curvatures time of full geometrodynamics). Such attempts suffer from the multiple choice problem, the spacetime problem (it is difficult to find a many-fingered internal time which is a spacetime scalar), and from the spectral analysis problem (the Hamiltonian of the theory is typically defined only implicitly by spectral analysis). Internal times proposed so far and likely to be contemplated in the future make the spectral analysis problem so involved and the true Hamiltonian so cumbersome that I find it hard to believe that the internal Schrödinger interpretation of full quantum gravity is a viable option.

The attempts to find an internal time among the matter variables have a hard time to cope with the energy conditions. The *matter time* provided by Gaussian reference dust comes as close to accomplishing the goal of the internal time framework as anyone can reasonably hope. It must be ultimately dismissed as a fundamental solution to the time problem only because it relies on a highly phenomenological system.

The last candidate for an internal time, the cosmological time of *unimodular gravity*, is not a many-fingered time variable, and hence it leaves an infinite number of remaining super-Hamiltonian constraints without dynamical interpretation.

The interpretations within the Wheeler-DeWitt framework suffer from the Hilbert space problem. All of them are also beset by the sandwich problem and

the spacetime problem. The *Klein-Gordon interpretation* fails to construct the Hilbert space because the super-Hamiltonian is not stationary in superspace and the potential term is not positive. The *semiclassical interpretation* hides the problem of time behind an approximation procedure. It claims that the Wheeler-DeWitt equation for a semiclassical state approximately reduces to the Schrödinger equation, and the Klein-Gordon norm reduces to the Schrödinger norm. Unfortunately, it achieves the positivity of the norm at an unacceptable price of suspending the superposition principle. Its attempts to exorcise the interference terms, by decoherence arguments or otherwise, are unconvincing. When one allows superpositions, one can show that the Schrödinger equation with its complex structure actually does not follow from the real Klein-Gordon equation, and one can easily exhibit states with a negative norm. Last but not least, when applying the semiclassical interpretation to quantum gravity proper, one again faces the problem of separating the classical modes that define time from the quantum modes.

A closer look at *third quantization* reveals that it does not really solve the problem of time and that its formalism resists an operational interpretation.

The last class of interpretations maintains that one does not need time to interpret quantum gravity. The *naive Schrödinger interpretation*, the *conditional probability interpretation*, and the *sum-over-histories interpretation* share the common Hilbert space, that of square-integrable functions on the configuration space(time) before the imposition of the constraints. With it they also share the same basic difficulties. The main one is that each of these interpretations in its own way violates the constraints. Further, none of them reproduces the standard results when applied to a Klein-Gordon system. In the case of the sum-over-histories interpretation, this casts doubts on the universal applicability of the probability amplitude rule. Finally, while the naive Schrödinger interpretation lacks dynamics, the conditional probability interpretation predicts wrong dynamics for well understood systems, like the parametrized Newtonian particle.

The *frozen time formalism* stands apart from the other three interpretations in this class. Its difficulty has always been how to explain quantum dynamics in terms of constants of the motion. Rovelli's proposal on how to achieve this goal is ambiguous. When its consequences are analyzed, it becomes apparent that it faces all the familiar difficulties, especially the Hilbert space problem and the multiple choice problem.

In my opinion, none of us has so far succeeded in proposing an interpretation of quantum gravity that would either solve or circumvent the problems of time. He who made eternity out of years remains beyond our reach. His ways remain unscrutable because He not only plays dice with matter but also with time.

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## 18. References

1. A. Gryphius, *Gesamtausgabe der deutschsprachigen Werke*, eds. M. Szyrocki and H. Powell (Niemeyer, Tübingen, 1964), Vol. 2, pp. 182-3.
2. R.F. Baierlein, D.H. Sharp and J.A. Wheeler, *Phys. Rev.* **126** (1962) 1864. "Three-Dimensional Geometry as Carrier of Information About Time."
3. K.V. Kuchař, *J. Math. Phys.* **13** (1972) 768. "Bubble-Time Canonical Formalism for Geometrodynamics."
4. A. Peres, *Nuovo Cimento* **26** (1962) 53. "On the Cauchy Problem in General Relativity II."
5. U. Gerlach, *Phys. Rev.* **177** (1969) 1929. "Derivation of the Ten Einstein Field Equations from the Semiclassical Approximation to Quantum Geometrodynamics."
6. C. Lanczos, *The Variational Principles of Mechanics*, 4th ed. (Univ. of Toronto Press, Toronto, 1970), Ch. VIII. 7, pp. 264-276.
7. Ref. 6, Ch. VIII. 6, pp. 262-264.
8. Ref. 6, Ch. XI. 10, pp. 185-192.
9. P. Hájíček, *Phys. Rev.* **D34** (1986) 1040. "Origin of Non-Unitarity in Quantum Gravity."
10. P. Hájíček, *J. Math. Phys.* **30** (1989) 2488. "Topology of Parametrized Systems."
11. P. Hájíček, *Class. Quantum Grav.* **7** (1990) 871. "Dirac Quantization of Systems with Quadratic Constraints."

12. M. Schön and P. Hájček, *Class. Quantum Grav.* 7 (1990) 861. "Topology of Quadratic Super-Hamiltonians."
13. P. Hájček, *Lecture Notes on Quantum Cosmology* (University of Bern, Bern, 1990); unpublished.
14. C. Carathéodory, *Variationsrechnung und partielle Differentialgleichungen erster Ordnung I* (Teubner, Leipzig, 1956).
15. J.L. Synge, in *Handbuch der Physik*, Vol. III/1, ed. S. Flügge (Springer, Berlin, 1960). "Classical Dynamics. 88. Generating Functions."
16. A. Peres and N. Rosen, *Nuovo Cimento* 13 (1959) 430. "On Cauchy's Problem in General Relativity."
17. A. Peres, *Nuovo Cimento* 18 (1960) 32. "Invariants of Relativity II. - Classification of Spaces."
18. J.A. Wheeler, in *Relativity, Groups, and Topology*, eds. C. and B. DeWitt (Gordon and Breach, New York, 1964). "Geometrodynamics and the Issue of the Final State."
19. J.A. Wheeler, in *Gravitation and Relativity*, eds. H -Y. Chiu and W.F. Hoffmann (Benjamin, New York, 1964). "Mach's Principle as Boundary Condition for Einstein's Equations."
20. E.P. Belasco and H. Ohanian, *J. Math. Phys.* 10 (1969) 1503. "Initial Conditions in General Relativity: Lapse and Shift Formulation."
21. P.G. Bergmann, in *Relativity*, eds. M. Carmeli, S. Fickler and L. Witten (Plenum, New York, 1980). "The Sandwich Conjecture."
22. A. Komar, *J. Math. Phys.* 11 (1970) 820. "General Relativistic Thin-Sandwich Theorem."
23. C.M. Pereira, *J. Math. Phys.* 14 (1973) 1498. "Thin-Sandwich Conjecture and a Fifth Initial Value Equation for the Gravitational Field."
24. D. Christodoulou and M. Francaviglia, in *Enrico Fermi Proceedings*, ed. J. Ehlers (North Holland, New York, 1979). "The Geometry of the Thin-Sandwich Problem."
25. C.M. Pereira, *J. Math. Phys.* 22 (1980) 1064. "Nonglobal Proof of the Thin-Sandwich Conjecture."
26. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva Univ., New York, 1964).
27. P.A.M. Dirac, *Can. J. Math.* 3 (1951) 1. "The Hamiltonian Form of Field Dynamics."
28. K.V. Kuchař and C.G. Torre, *Phys. Rev. D* 43 (1990) 419. "Gaussian Reference Fluid and Interpretation of Quantum Geometrodynamics."
29. C. W. Misner, *Rev. Mod. Phys.* 29 (1957) 497. "Feynman Quantization of General Relativity."
30. P.W. Higgs, *Phys. Rev. Lett.* 1 (1958) 373. "Integration of Secondary Constraints in General Relativity."

31. J.A. Wheeler, in *Batelle Rencontres: 1967 Lectures in Mathematics and Physics*, eds. C. DeWitt and J.A. Wheeler (Benjamin, New York, 1968). "Superspace and the Nature of Geometrodynamics."
32. B.S. DeWitt, *Phys Rev.* **160** (1967) 1113. "Quantum Theory of Gravity I."
33. K.V. Kuchař, in *Quantum Gravity 2: A Second Oxford Symposium*, eds. C.J. Isham, R. Penrose and D.W. Sciama (Clarendon, Oxford, 1981). "Canonical Methods of Quantization."
34. C.J. Isham, in *Relativity, Groups and Topology II: Les Houches 1983*, eds. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984). "Topological and Global Aspects of Quantum Theory."
35. K.V. Kuchař, *Phys. Rev.* **D4** (1971) 955. "Canonical Quantization of Cylindrical Gravitational Waves."
36. R.H. Gowdy, *Ann. Phys. (N.Y.)* **83** (1974) 203. "Vacuum Spacetimes with Two-Parameter Spacelike Isometry Groups and Compact Invariant Hypersurfaces: Topologies and Boundary Conditions."
37. C.W. Misner, *Phys. Rev.* **D8** (1973) 3271. "A Minisuperspace Example - The Gowdy  $T^3$  Universe."
38. B. Berger, *Phys. Rev.* **D11** (1975) 2770. "Quantum Cosmology: Exact Solution for the Gowdy  $T^3$  Model."
39. B. Berger, *Phys. Lett.* **B108** (1982). "Singularity Avoidance in the Semiclassical Gowdy  $T^3$  Model."
40. B. Berger, *Ann. Phys. (N.Y.)* **156** (1984) 155. "Quantum Effects in the Gowdy  $T^3$  Cosmology."
41. V. Husain, *Class. Quantum Grav.* **4** (1987) 1587. "Quantum Effects on the Singularity of the Gowdy Cosmology."
42. M.O. Green, J. Schwarz and E. Witten, *Superstring Theory* (Cambridge Univ. Press, Cambridge, 1987).
43. L. Brink and M. Henneaux, *Principles of String Theory* (Plenum, New York, 1988).
44. K.V. Kuchař and C.G. Torre, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991). "Strings as Poor Relatives of General Relativity."
45. P.A.M. Dirac, *Can. J. Math.* **3** (1951) 1. "The Hamiltonian Form of Field Dynamics."
46. K.V. Kuchař, *J. Math. Phys.* **17** (1976) 792. "Kinematics of Tensor Fields in Hyperspace."
47. K.V. Kuchař, *J. Math. Phys.* **17** (1976) 801. "Dynamics of Tensor Fields in Hyperspace."
48. J.J. Halliwell and S.W. Hawking, *Phys. Rev.* **D31** (1985) 1777. "Origin of Structure in the Universe."

49. C.W. Misner, in *Magic Without Magic: John Archibald Wheeler, A Collection of Essays in Honor of his 60th Birthday*, ed. J. Klauder (Freeman, San Francisco, 1972). "Minisuperspace."
50. M. Ryan, *Hamiltonian Cosmology* (Springer, Berlin, 1972).
51. M.A.H. MacCallum, in *Quantum Gravity*, eds. C.J. Isham, R. Penrose and D. W. Sciama (Clarendon, Oxford, 1975). "Quantum Cosmological Models."
52. J.J. Halliwell, *Int. J. Mod. Phys. A*5 (1990) 2473. "A Bibliography of Papers on Quantum Cosmology."
53. C.W. Misner, *Phys. Rev. Lett.* 22 (1969) 1071. "Mixmaster Universe."
54. C.W. Misner, *Phys. Rev.* 186 (1969) 1319, 1328. "Quantum Cosmology."
55. C.W. Misner, in *Relativity*, eds. M. Carmeli, S. Fickler and L. Witten (Plenum, New York, 1970). "Classical and Quantum Dynamics of a Closed Universe."
56. J. Abbot, S. Giddings and K.V. Kuchař, *Gen. Rel. Grav.* 16 (1984) 751. "Einstein's Theory in a Three-Dimensional Spacetime."
57. E. Witten, *Nucl. Phys.* B311 (1988) 46. "2+1 Dimensional Gravity as an Exactly Soluble System."
58. E. Witten, *Nucl. Phys.* B323 (1989) 113. "Topology-Changing Amplitudes in 2+1 Dimensional Gravity."
59. V. Moncrief, *J. Math. Phys.* 30 (1989) 2907. "Reduction of the Einstein Equations in 2+1 Dimensions to a Hamiltonian System over Teichmüller Space."
60. A. Hosoya and K. Nakao, *Class. Quantum Grav.* 7 (1990) 163. "(2+1)-Dimensional Pure Gravity for an Arbitrary Closed Initial Surface."
61. S. Carlip, *Phys. Rev.* D42 (1990) 2647. "Observables, Gauge Invariance, and Time in (2+1)-Dimensional Quantum Gravity."
62. S. Carlip, *Preprint UCDPHYS-PUB-23-90*. "Time in 2+1 Dimensional Quantum Gravity."
63. G. Nordström, *Phys. Z.* 13 (1912) 1126. "Relativitätsprinzip und Gravitation."
64. G. Nordström, *Ann. Phys. (Leipzig)* 42 (1913) 533. "Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips."
65. A. Einstein and A.D. Fokker, *Ann. Phys. (Leipzig)* 44 (1914) 321. "Die Nordströmsche Gravitationstheorie vom Standpunkt des Absoluten Differentialkalküls."
66. P. Hájíček and K.V. Kuchař, *Phys. Rev.* D41 (1990) 1091. "Constraint Quantization of Parametrized Relativistic Gauge Systems in Curved Spacetimes."



67. K.V. Kuchař, *Phys. Rev. D* **22** (1980) 1285. "Gravitation, Geometry, and Nonrelativistic Quantum Theory."
68. J.B. Hartle and K.V. Kuchař, *J. Math. Phys.* **25** (1984) 57. "Path Integrals in Parametrized Theories: Newtonian Systems."
69. J.B. Barbour, *Nature* **249** (1974) 328. "Relative-Distance Machian Theories."
70. J.B. Barbour, *Nature* **250** (1974) 606. "Erratum."
71. J.B. Barbour, *Nuovo Cimento* **B26** (1975) 16. "Forceless Machian Dynamics."
72. J.B. Barbour and B. Bertotti, *Nuovo Cimento* **B38** (1977) 1. "Gravity and Inertia in a Machian Framework."
73. J.B. Barbour and B. Bertotti, *Proc. Roy. Soc. London* **A382** (1982) 295. "Mach's Principle and the Structure of Dynamical Theories."
74. W. Pauli, in *Handbuch d. Physik V*, ed. S. Flugge (Springer, Berlin, 1958). "Die allgemeinen Prinzipien der Wellenmechanik."
75. J.B. Hartle, *Phys. Rev. D* **38** (1988) 2985. "Quantum Kinematics of Spacetime II. A Model Quantum Cosmology with Real Clocks."
76. W.G. Unruh and R.M. Wald, *Phys. Rev. D* **40** (1989) 2598. "Time and the Interpretation of Canonical Quantum Gravity."
77. K.V. Kuchař, *Phys. Rev. D* **39** (1989) 1579. "Parametrized Scalar Field on  $R \times S^1$ : Dynamical Pictures, Spacetime Diffeomorphisms and Conformal Isometries."
78. K.V. Kuchař, *Phys. Rev. D* **39** (1989) 2263. "The Dirac Constraint Quantization of a Parametrized Field Theory by Anomaly-Free Operator Representations of Spacetime Diffeomorphisms."
79. K.V. Kuchař, in *Highlights in Gravitation and Cosmology*, eds. B.R. Iyer et al. (Cambridge Univ. Press, Cambridge, 1988). "Canonical Quantization of Generally Covariant Systems."
80. G. Torre, *Class. Quantum Grav.* **8** (1991) 1895. "A Complete Set of Observables for Cylindrically Symmetric Gravitational Fields."
81. K.V. Kuchař and C.G. Torre, *J. Math. Phys.* **30** (1989) 1769. "World Sheet Diffeomorphisms and the Canonical String."
82. D.G. Boulware and S. Deser, *J. Math. Phys.* **8** (1967) 1468. "Stress-Tensor Commutators and Schwinger Terms."
83. J. Anderson, *Phys. Rev.* **114** (1959) 1182. "Factor Sequences in Quantized General Relativity."
84. J. Anderson, in *Proceedings of the First Eastern Theoretical Physics Conference, October 26-27, 1962*, ed. M.E. Ross (Gordon and Breach, New York, 1962). "Q-Number Coordinate Transformations and the Ordering Problem in General Relativity."

85. J. Schwinger, *Phys. Rev.* **132** (1963) 1317. "Quantized Gravitational Field II."
86. P.A.M. Dirac, in *Contemporary Physics: Trieste Symposium 1968*, V. 1, ed. L. Fonda (IAEA, Vienna, 1969). "The Quantization of the Gravitational Field."
87. A. Komar, *Phys. Rev.* **D19** (1979) 2908. "Constraints, Hermiticity, and Correspondence."
88. A. Komar, *Phys. Rev.* **D20** (1979) 830. "Consistent Factor Ordering of General-Relativistic Constraints."
89. A. Ashtekar, *Phys. Rev.* **D36** (1987) 1587. "New Hamiltonian Formulation of General Relativity."
90. N.C. Tsamis and R.P. Woodard, *Phys. Rev.* **D36** (1987) 3641, "The Factor-Ordering Problem Must be Regulated."
91. J.L. Friedman and I. Jack, *Phys. Rev.* **D37** (1988) 3495. "Formal Commutators of the Gravitational Constraints are not Well-Defined: A Translation of Ashtekar's Ordering to the Schrödinger Representation."
92. J.L. Friedman and I. Jack, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991). "Formal Commutators of the Gravitational Constraints are not Well-Defined."
93. P. Hájíček and K.V. Kuchař, *J. Math. Phys.* **31** (1990) 1723. "Transversal Affine Connection and Quantization of Constrained Systems."
94. P.A.M. Dirac, *Phys. Rev.* **114** (1959) 924. "Fixation of Coordinates in the Hamiltonian Theory of Gravitation."
95. K.V. Kuchař, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991). "The Problem of Time in Canonical Quantization of Relativistic Systems."
96. C.J. Isham, *Imperial College Preprint* TP/90-91/14. "Conceptual and Geometrical Problems in Quantum Gravity. Lectures Presented at the 1991 Schladming Winter School."
97. J.B. Barbour and L. Smolin, *Yale University Preprint* April 8, 1988. "Can Quantum Mechanics be Sensibly Applied to the Universe as a Whole?"
98. J.J. Halliwell, *Phys. Rev.* **D43** (1991) 2590. "Global Spacetime Symmetries in the Functional Schrödinger Picture."
99. R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962). "The Dynamics of General Relativity."
100. R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **117** (1960) 1595. "Canonical Variables for General Relativity."
101. K.V. Kuchař, *J. Math. Phys.* **11** (1970) 3322. "Ground State Functional of the Linearized Gravitational Field."

102. K.V. Kuchař, in *Relativity, Astrophysics and Cosmology*, ed. W. Israel (Reidel, Dordrecht, 1973). "Canonical Quantization of Gravity."
103. J.W. York, *Phys. Rev. Lett.* **28** (1972) 1082. "Role of Conformal Three-Geometry in the Dynamics of Gravitation."
104. J.W. York, in *Sources of Gravitational Radiation*, ed. L. Smarr (Cambridge Univ. Press, Cambridge, 1979). "Kinematics and Dynamics of General Relativity."
105. Y. Choquet-Bruhat and J.W. York, in *General Relativity and Gravitation the Hundred Years After the Birth of Albert Einstein*, eds. P. Bergmann, J. Goldberg and A. Held (Plenum, New York, 1980). "The Cauchy Problem."
106. I.M. Singer, *Commun. Math. Phys.* **60** (1978) 7. "Some Remarks on the Gribov Ambiguity."
107. D.N. Page, in *Quantum Concepts in Space and Time*, eds. R. Penrose and C.J. Isham (Clarendon, Oxford, 1986). "Hawking's Wave Function for the Universe."
108. Y. Choquet-Bruhat, *C.R. Acad. Sci. (Paris)* **280A** (1975) 169. "Sous variétés maximales, ou à courbure constant, de variétés lorentziennes."
109. D. Brill and F. Flaherty, *Commun. Math. Phys.* **50** (1976) 157. "Isolated Maximal Surfaces in Spacetime."
110. A.J. Goddard, *Commun. Math. Phys.* **54** (1977) 279. "Foliations of Space-Times by Spacelike Hypersurfaces of Constant Mean Curvature."
111. L.L. Smarr and J.W. York, *Phys. Rev.* **D17** (1978) 2529. "Kinematical Conditions in the Construction of Spacetime."
112. D.M. Eardley and L.L. Smarr, *Phys. Rev.* **D19** (1979) 2239. "Time Functions in Numerical Relativity: Marginally Bound Dust Collapse."
113. J.E. Marsden and F.J. Tipler, *Physics Rep.* **66** (1980) 109. "Maximal Hypersurfaces and Foliations of Constant Mean Curvature in General Relativity."
114. J. Isenberg and J.E. Marsden, *J. Geom. and Phys.* **1** (1984) 85. "The York Map is a Canonical Transformation."
115. W. Blyth and C.J. Isham, *Phys. Rev.* **D11** (1975) 768. "Quantization of a Friedman Universe Filled with a Scalar Field."
116. K.V. Kuchař, *J. Math. Phys.* **17** (1976) 777, Section 10. "Geometry of Hyperspace."
117. K.V. Kuchař, *J. Math. Phys.* **17** (1976) 792, Section 3. "Kinematics of Tensor Fields in Hyperspace."
118. A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie* (Vieweg, Braunschweig, 1920), §28, p.67; English transl. R.W. Lawson, *Relativity: The Special and the General Theory*, 17th ed. (Crown, New York, 1961).

119. D. Hilbert, *2 Mitt., Nachr. Ges. Wiss. Göttinger* 53 (1917). "Grundlagen der Physik." Included in *David Hilbert gessammelte Abhandlungen*, Vol. 3, 2nd ed. (Springer, Berlin, 1970).
120. B.S. DeWitt, in *Gravitation: An Introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962). "The Quantization of Geometry."
121. N. Bohr and L. Rosenfeld, *Mat.-Fys. Medd. Dan. vid. Selsk.* 12 No. 8 (1933). "Zur Frage der Messbarkeit der elektromagnetischen Feldgrößen." English transl. in *Selected Papers by Léon Rosenfeld*, eds. R.S. Cohen and J. Stachel, *Boston Studies in the Philosophy of Science* 21 (Reidel, Dordrecht, 1978). "On the Question of the Measurability of Electromagnetic Field Quantities."
122. C. Rovelli, *Sissa Preprint* 101/89/ep. (1989). "What is Observable in Classical and Quantum Gravity."
123. P.G. Bergmann and A. Komar, *Int. J. Theor. Phys.* 5 (1972) 15. "The Coordinate Group of Symmetries of General Relativity."
124. D.C. Salisbury and K. Sundermeyer, *Phys. Rev.* D27 (1983) 740. "Realization in Phase Space of General Coordinate Transformations."
125. J. Lee and R.M. Wald, *J. Math. Phys.* 31 (1990) 725. "Local Symmetries and Constraints."
126. R.M. Wald, *Univ. of Chicago Preprint* (1991). "The Relationship between Local Symmetries and Constraints in General Relativity and Other Classical Lagrangian Field Theories."
127. C.J. Isham and K.V. Kuchař, *Ann. Phys. (N.Y.)* 164 (1985) 288. "Representations of Spacetime Diffeomorphisms. I. Canonical Parametrized Field Theories."
128. C.J. Isham and K.V. Kuchař, *Ann. Phys. (N.Y.)* 164 (1985) 316. "Representations of Spacetime Diffeomorphisms. II. Canonical Geometrodynamics."
129. K.V. Kuchař, *Found. Phys.* 16 (1986) 193. "Canonical Geometrodynamics and General Covariance."
130. J.B. Hartle, in *Proceedings of the Fifth Marcel Grossmann Meeting*, eds. D. Blair and M. Buckingham (World Scientific, Singapore, 1989). "Time and Prediction in Quantum Cosmology."
131. J.J. Halliwell and J.B. Hartle, *Phys. Rev.* D43 (1991) 1170. "Wave Functions Constructed from Invariant Sums-Over-Histories Satisfy Constraints."
132. K.V. Kuchař and C.G. Torre, *Phys. Rev.* D44 (1991); in print. "Harmonic Gauge in Canonical Gravity."
133. C.L. Stone and K.V. Kuchař, submitted to *Class. Quantum Grav.* "Representations of Spacetime Diffeomorphisms in Canonical Geometrodynamics under Harmonic Coordinate Conditions."

134. K.V. Kuchař, submitted to *Phys. Rev. D*. "Extrinsic Curvature as a Reference Fluid in Canonical Gravity."
135. C.G. Torre, *Phys. Rev. D* **40** (1989) 2588. "Hamiltonian Formulation of Induced Gravity in Two Dimensions."
136. K.K. Tam and J. O'Hanlon, *Nuovo Cimento* **62B** (1969) 351 "Relativistic Magnetohydrodynamics of a Gravitating Fluid."
137. B.F. Schutz, *Phys. Rev. D* **4** (1971) 3559. "Hamiltonian Theory of a Relativistic Perfect Fluid."
138. C. Eckart, *Phys. Rev.* **58** (1940) 919. "The Thermodynamics of Irreversible Processes III. Relativistic Theory of the Single Fluid."
139. G.A. Kluitenberg and S.R. deGroot, *Physica* **21** (1955) 148. "Relativistic Thermodynamics of Irreversible Processes. IV. Systems with Polarization and Magnetization in an Electromagnetic Field."
140. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, 1973) Section 4.3.
141. Ref 28, Sections 8 and 10.
142. A. Einstein, *Ann. Phys. (Leipzig)* **49** (1916) 769; see pp. 801 and 812. "Die Grundlage der allgemeinen Relativitätstheorie."
143. W.G. Unruh, *Phys. Rev. D* **40** (1989) 1048. "Unimodular Theory of Canonical Quantum Gravity."
144. A. Einstein, *Sitzungsber. Preuss. Acad. Wiss., Phys. - Math. Kl.* (1919) English transl. in *The Principle of Relativity* (Dover, New York, 1952). "Do Gravitational Fields Play an Essential Part in the Structure of the Elementary Particles of Matter?"
145. J.J. Van de Bij, H. van Dam and Y.J. Ng, *Physica A* **116** (1982) 307 "The Exchange of Massless Spin-Two Particles."
146. F. Wilczek, *Phys. Rep.* **104** (1984) 143. "Foundations and Working Pictures in Microphysical Cosmology."
147. W. Buchmüller and N. Dragon, *Phys. Lett.* **B207** (1988) 292. "Einstein Gravity from Restricted Coordinate Invariance."
148. S. Weinberg, *Rev. Mod. Phys.* **61** (1989) 1. "The Cosmological Constant Problem."
149. M. Henneaux and C. Teitelboim, *Phys. Lett.* **B222** (1989) 195. "The Cosmological Constant and General Covariance."
150. W.G. Unruh, in *Proceedings of the Fourth Seminar on Quantum Gravity*, eds. M.A. Markov, V.A. Berezin and V.P. Frolov (World Scientific, Singapore, 1988). "Time and Quantum Gravity."
151. R.D. Sorkin, in *History of Modern Gauge Theories*, eds. M. Dresden and A. Rosenblum (World Scientific, Singapore, 1990). "On the Role of Time in the Sum-Over-Histories Framework for Gravity."

152. R.D. Sorkin, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991). "Problems with Causality in the Sum-Over-Histories Framework for Quantum Mechanics."
153. J.D. Brown and J.W. York, *Phys. Rev. D* **40** (1989) 3312. "Jacobi's Action and the Recovery of Time in General Relativity."
154. K.V. Kuchař, *Phys. Rev. D* **43** (1991) 3332. "Does an Unspecified Cosmological Constant Solve the Problem of Time in Quantum Gravity?"
155. A. Ashtekar and A. Magnon, *Proc. Roy. Soc. London A* **346** (1975) 375. "Quantum Fields in Curved Spacetimes."
156. A. Magnon-Ashtekar, *Thesis* (Département de Mathématiques Pures, Université de Clermont, Clermont, 1976); unpublished.
157. A. Ashtekar and A. Magnon-Ashtekar, *C.R. Acad. Sci. (Paris)* **A286** (1978) 531. "Remark on Quantum Field Theory in Curved Space-Times."
158. B.S. Kay, *Commun. Math. Phys.* **62** (1978) 55. "Linear Spin-Zero Quantum Fields in External Gravitational and Scalar Fields."
159. K.V. Kuchař, *J. Math. Phys.* **22** (1981) 2640. "General Relativity: Dynamics Without Symmetry."
160. K.V. Kuchař, *J. Math. Phys.* **25** (1982) 1647. "Conditional Symmetries in Parametrized Field Theories."
161. J. Friedman and A. Higuchi, *Phys. Rev. D* **41** (1990) 2479. "Symmetry and Internal Time on the Superspace of Asymptotically Flat Geometries."
162. K.V. Kuchař, *J. Math. Phys.* **18** (1977) 1589. "Geometrodynamics with Tensor Sources."
163. T.P. Singh and T. Padhanabhan, *Ann. Phys. (N.Y.)* **196** (1989) 296. "Notes on Semiclassical Gravity."
164. T. Banks, *Nucl. Phys. B* **249** (1985) 332. "TCP, Quantum Gravity, the Cosmological Constant, and All That."
165. A. Vilenkin, *Phys. Rev. D* **39** (1989) 1116. "The Interpretation of the Wave Function of the Universe."
166. R. Brout, *Found. Phys.* **17** (1987) 603. "On the Concept of Time and the Origin of the Cosmological Temperature."
167. R. Brout and G. Venturi, *Phys. Rev. D* **39** (1989) 2436. "Time in Semiclassical Gravity."
168. R. Brout, G. Horowitz, and D. Wiel, *Phys. Lett. B* **192** (1987) 318. "On the Onset of Time and Temperature in Cosmology."
169. J.B. Hartle, in *Gravitation in Astrophysics, Cargese, 1986*, eds. B. Carter and J.B. Hartle (Plenum, New York, 1986). "Prediction and Observation in Quantum Cosmology."
170. J.J. Halliwell, in *Quantum Cosmology and Baby Universes*, eds. S. Coleman, J. Hartle, T. Piran and S. Weinberg (World Scientific, Singapore, 1991). "Introductory Lectures on Quantum Cosmology."

171. T. Padhanabhan, *Class. Quantum Grav.* **6** (1989) 533. "Semiclassical Approximations to Gravity and the Issue of Back-Reaction."
172. C. Møller, in *Les Théories Relativistes de la Gravitation*, eds. A. Lichnerowicz and M.A. Tonnelat (CNRS, Paris, 1962). "The Energy-Momentum Complex in General Relativity and Related Problems."
173. L. Rosenfeld, *Nucl. Phys.* **40** (1963) 353. "On Quantization of Fields."
174. H.D. Zeh, *Phys. Lett.* **A116** (1986) 9. "Emergence of Time From a Universal Wave Function."
175. H.D. Zeh, *Phys. Lett.* **A126** (1988) 311. "Time in Quantum Gravity."
176. E. Joos, *Phys. Lett.* **A116** (1986) 6. "Why do We Observe a Classical Spacetime?"
177. C. Kiefer, *Class. Quantum Grav.* **4** (1987) 1369. "Continuous Measurement of Mini-Superspace Variables by Higher Multipoles."
178. J.J. Halliwell, *Phys. Rev.* **D39** (1989) 2912. "Decoherence in Quantum Cosmology."
179. V. Lapchinski and V.A. Rubakov, *Acta Phys. Pol.* **B10** (1979) 1041. "Canonical Quantization of Gravity and Quantum Field Theory in Curved Spacetime."
180. J.J. Halliwell, *Phys. Rev.* **D36** (1987) 3626. "Correlations in the Wave Function of the Universe."
181. J.B. Barbour, *Preprint 1991*. "Time, Gauge Fields, and the Schrödinger Equation in Quantum Cosmology."
182. A. Vilenkin, *Phys. Rev.* **D37** (1988) 888. "Quantum Cosmology and the Initial State of the Universe."
183. T. Padhanabhan, *Pramana J. Phys.* **35** (1990) L199. "A Definition for Time in Quantum Cosmology."
184. J. Greensite, *Nucl. Phys.* **B342** (1990) 409. "Time and Probability in Quantum Cosmology."
185. C. Kiefer, *Zürich University Preprint ZU-TH 10/91*. To appear in the *Proceedings of the Journées Relativistes, Cargèse, 2.5.-4.5 1991*. "How does Quantum Gravity Modify the Schrödinger Equation for Matter Fields?" C. Kiefer and T.P. Singh, *Preprint 1990*. "Quantum Gravitational Corrections to the Functional Schrödinger Equation."
186. G.T. Horowitz and R.M. Wald, *Phys. Rev.* **D21** (1980) 1462. "Quantum Stress Energy in Nearly Conformally Flat Spacetimes."
187. S. Danjbar-Daemi, B.S. Kay and T.W.B. Kibble, *Phys. Lett.* **91B** (1980) 417. "Renormalization of Semiclassical Field Theories."
188. T.W.B. Kibble, in *Quantum Gravity 2: A Second Oxford Symposium*, eds. C.J. Isham, R. Penrose and D.W. Sciama (Clarendon, Oxford, 1981). "Is Semiclassical Theory of Gravity Viable?"

189. M.J. Duff, in *Quantum Gravity 2: A Second Oxford Symposium*, eds. C.J. Isham, R. Penrose and D. W. Sciama (Clarendon, Oxford, 1981) "Inconsistency of Quantum Field Theory in Curved Space-Time."
190. W-M. Suen, *Phys. Rev. Lett.* **62** (1989) 2217. "Minkowski Spacetime is Unstable in Semiclassical Gravity."
191. W-M. Suen, *Phys. Rev.* **D40** (1989) 315. "Stability of the Semiclassical Einstein Equation."
192. S. Giddings and A. Strominger, *Nucl. Phys.* **B321** (1988) 481. "Baby Universes, Third Quantization and the Cosmological Constant."
193. S. Coleman, *Nucl. Phys.* **B310** (1988) 643. "Why is There Something Rather Than Nothing: A Theory of the Cosmological Constant."
194. M. McGuigan, *Phys. Rev.* **D38** (1988) 3031. "Third Quantization and the Wheeler-DeWitt Equation."
195. M. McGuigan, *Phys. Rev.* **D39** (1989) 2229. "Universe Creation From the Third Quantized Vacuum."
196. L. Rosenfeld, in *Niels Bohr and the Development of Physics* (Pergamon, London, 1955). "On Quantum Electrodynamics."
197. C. Møller, *The Theory of Relativity*, 2nd ed. (Clarendon, Oxford, 1972). "8.8. Space and Time Measurements in an Arbitrary System of Reference."
198. L.D. Landau and E.M. Lifshitz, *Teorija polja* (Gosudarstvennoe izdatel'stvo fizikomatematicheskoy literatury, Moskva, 1962). English transl. M. Hamermesh, *The Classical Theory of Fields* (Pergamon, Oxford, 1962). "84. Distances and Time Intervals."
199. J.B. Hartle and S.W. Hawking, *Phys. Rev.* **D28** (1983) 2960. "Wave Function of the Universe."
200. S.W. Hawking, *Nucl. Phys.* **B239** (1984) 257. "The Quantum State of the Universe."
201. S.W. Hawking, in *Relativity, Groups and Topology II, Les Houches Session XL*, eds. B. DeWitt and R. Stora (North Holland, Amsterdam, 1984). "Lectures on Quantum Cosmology."
202. S.W. Hawking and D.N. Page, *Nucl. Phys.* **B264** (1986) 185. "Operator Ordering and the Flatness of the Universe."
203. S.W. Hawking and D.N. Page, *Nucl. Phys.* **B298** (1988) 789. "How Probable is Inflation?"
204. D.N. Page, *Phys. Rev.* **D34** (1986) 2267. "Density Matrix of the Universe."
205. D.N. Page and W.K. Wootters, *Phys. Rev.* **D27** (1983) 2885. "Evolution Without Evolution: Dynamics Described by Stationary Observables."
206. D.N. Page, *ITP Preprint* NSF-ITP-89-18 (1989). "Time as an Inaccessible Observable."



207. D.N. Page, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991). "Interpreting the Density Matrix of the Universe."
208. W.K. Wootters, *Int. J. Theor. Phys.* **23** (1984) 701. "Time Replaced by Quantum Correlations."
209. C. Teitelboim, *Phys. Lett.* **96B** (1980) 77. "Proper Time Approach to the Quantization of the Gravitational Field."
210. C. Teitelboim, *Phys. Rev.* **D25** (1982) 3159. "Quantum Mechanics of the Gravitational Field."
211. M. Henneaux and C. Teitelboim, *Ann. Phys.* **143** (1982) 127. "Relativistic Quantum Mechanics of Supersymmetric Particles."
212. J.B. Hartle, *Phys. Rev.* **D37** (1988) 2818. "Quantum Kinematics of Spacetime I: Non-Relativistic Theory." *Phys. Rev.* **D43** (1991) 1434. "Erratum."
213. J.B. Hartle, *VCSB Preprint*. "Quantum Kinematics of Spacetime III: General Relativity."
214. J.B. Hartle, in *The Proceedings of the 60th Birthday Celebration for Murray Gell-Mann, Pasadena, CA, January 27, 1989*, eds. J. Schwarz and F. Zachariasen (Cambridge Univ. Press, Cambridge, 1990). "Excess Baggage."
215. M. Gell-Mann and J.B. Hartle, in *Complexity, Entropy, and the Physics of Information, Santa Fe Institute Studies in the Sciences of Complexity, VIII*, ed. W.H. Zurek (Addison-Wesley, Reading, 1990). "Quantum Mechanics in the Light of Quantum Cosmology."
216. J.B. Hartle, in *Quantum Cosmology and Baby Universes*, eds. S. Coleman, J. Hartle, T. Piran and S. Weinberg (World Scientific, Singapore, 1991). "The Quantum Mechanics of Cosmology."
217. J.B. Hartle, *Preprint UCSBTH91-02*. "Spacetime Coarse Grainings in Non-Relativistic Quantum Mechanics."
218. P.G. Bergmann, *Nuovo Cimento* **3** (1956) 1956. "Introduction of 'True Observables' into the Quantum Field Equations."
219. P.G. Bergmann, *Helv. Phys. Acta Suppl.* **4** (1956) 1956. "Quantisierung allgemeinkovarianter Feldtheorien."
220. E. Newman and P.G. Bergmann, *Rev. Mod. Phys.* **29** (1957) 443. "Observables in Singular Theories by Systematic Approximation."
221. A. Komar, *Phys. Rev.* **111** (1958) 1182. "Construction of a Complete Set of Independent Observables in the General Theory of Relativity."
222. B.S. DeWitt, Ref. 32, Section 4, p. 160.
223. D.R. Brill and R.H. Gowdy, *Rep. Progress Phys.* **33** (1970) 413. "Quantization of General Relativity. 7.1. The Frozen Formalism Problem of Generally Covariant Theories."

224. C. Rovelli, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991). "Is There Incompatibility Between the Ways Time is Treated in General Relativity and in Standard Quantum Mechanics?"
225. C. Rovelli, *Phys. Rev. D* **42** (1990) 2638. "Quantum Mechanics Without Time: A Model."
226. C. Rovelli, *Phys. Rev. D* **43** (1991) 442. "Time in Quantum Gravity: An Hypothesis."
227. P. Hájíček, *Phys. Rev. D* **44** (1991) 1337. "Comment on 'Time in Quantum Gravity: An Hypothesis'."
228. L. Van Hove, *Acad. Roy. Belg. Bull. Cl. Sci.* **37** (1951) 610. "Sur le problème des relations entre les transformations unitaires de la mécanique quantique et les transformations canoniques de la mécanique classique."
229. C. Rovelli, *Phys. Rev. D* **44** (1991) 1339. "Quantum Evolving Constants. Reply to 'Comment on Time in Quantum Gravity: An Hypothesis'."
230. J.B. Hartle, *Private communication* (1991).