# Gravitational Energy-momentum and Conservation of Energy-momentum in General Relativity

Zhaoyan Wu
Center for Theoretical Physics, Jilin University, Jilin 130012, China

November 3, 2018

**Abstract**

Based on a general variational principle, Einstein-Hilbert action and sound facts from geometry, it is shown that the long existing pseudotensor, non-localizability problem of gravitational energy-momentum is a result of mistaking different geometrical, physical objects as one and the same.

It is also pointed out that in a curved spacetime, the sum vector of matter energy-momentum over a finite hyper-surface can not be defined. In curvilinear coordinate systems conservation of matter energy-momentum is not the continuity equations for its components. Conservation of matter energy-momentum is the vanishing of the covariant divergence of its density-flux tensor field. Introducing gravitational energy-momentum to save the law of conservation of energy-momentum is unnecessary and improper.

After reasonably defining "change of a particle's energy-momentum", we show that gravitational field does not exchange energy-momentum with particles. And it does not exchange energy-momentum with matter fields either. Therefore, the gravitational field does not carry energy-momentum, it is not a force field and gravity is not a natural force.

## 1 Motivation

The law of conservation of energy-momentum is the cornerstone of modern physics. But no sooner had Einstein established his theory of general relativity (GR), than he noticed that his field equation did not lead to a continuity equation for the conservation of matter energy-momentum he expected:

$$\int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) = 0, \forall\mu = 0, 1, 2, 3 \tag{1}$$

where $\Omega$ is an arbitrary region in spacetime $M$. In fact, applying the contracted Bianchi identity to Einstein's field equation

$$R^{\lambda\mu}(x) - \frac{1}{2}Rg^{\lambda\mu}(x) = \frac{8\pi G}{c^4}T^{\lambda\mu}(x), \forall\lambda,\mu = 0,1,2,3 \tag{2}$$

one gets

$$\nabla_\lambda T^{\lambda\mu}(x)|_p = 0, \forall\mu = 0,1,2,3; p \in M \tag{3}$$

Multiplying it by $\sqrt{-|g(x)|}$,

$$\frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}T^{\lambda\mu}(x)] + \sqrt{-|g(x)|}\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0 \tag{4}$$

and integrating eqn.(4) over $\Omega$, one gets

$$\int_{\partial\Omega} ds^\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) = -\int_\Omega d^4x\sqrt{-|g(x)|}\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x), \forall\mu = 0,1,2,3 \tag{5}$$

It is not eqn.(1). In order to save the law of conservation of energy-momentum in GR, Einstein recast eqn.(4) into the following form[1]

$$\frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}(T^{\lambda\mu}(x) + t^{\lambda\mu}(x))] = 0, \forall\mu = 0,1,2,3 \tag{6}$$

where $t^{\lambda\mu}(x)$ was interpreted as the gravitational energy-momentum. Integrating eqn.(6) over spactime domain $\Omega$, one gets by using Gauss theorem

$$\int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}(T^{\lambda\mu}(x) + t^{\lambda\mu}(x)) = 0, \forall\mu = 0,1,2,3 \tag{7}$$

which was taken for the continuity equation for conservation of total (matter plus gravitational) energy-momentum. In fact, when the boundary of $\Omega$ is composed of a past spacelike hyper-surface $\Sigma$, a future spacelike hyper-surface $\Sigma'$ and a timelike hyper-surface $\Gamma$ which links the boundaries of $\Sigma$ and $\Sigma'$, eqn.(7) can be written as

$$\int_{\Sigma'} ds_\lambda(x)\sqrt{-|g(x)|}(T^{\lambda\mu}(x) + t^{\lambda\mu}(x)) - \int_\Sigma ds_\lambda(x)\sqrt{-|g(x)|}(T^{\lambda\mu}(x) + t^{\lambda\mu}(x))$$
$$+ \int_\Gamma ds_\lambda(x)\sqrt{-|g(x)|}(T^{\lambda\mu}(x) + t^{\lambda\mu}(x)) = 0, \forall\mu = 0,1,2,3$$

This was read as, the difference between the matter plus gravitational energy-momentums on $\Sigma'$ and on $\Sigma$ equals the matter plus gravitational energy-momentum which flows in through $\Gamma$. Evidently, here the expression $ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)$ at the left hand side of the above equation has been taken as the $\mu$-component of matter energy-momentum on the small hyper-surface element at $x$, and $\int_\Sigma ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)$ has been taken as the $\mu$-component of matter energy-momentum 4-vector on $\Sigma$. The former is correct, however, the latter is wrong, since the coordinate system $\{x^0, x^1, x^2, x^3\}$ is curvilinear, and there is no flat coordinate system in the curved spacetime $M$ in GR. As a matter of fact, we can not define the sum vector of matter energy-momentum on a finite hyper-surface in the curved spacetime $M$ (See Section 2). We see, from the very beginning, introducing gravitational energy-momentum $t^{\lambda\mu}(x)$ to save the law of conservation of energy-momentum is based on taking $\int_\Sigma ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)$ for the $\mu$-component of the sum matter energy-momentum 4-vector on spacelike hyper-surface $\Sigma$, which does not really exist in curved spacetime. And this is not just a peripheral error.

Bauer immediately pointed out that this $t^{\lambda\mu}(x)$ is not a tensor, and is not localizable[2]. Besides, it is not symmetrical. When there is no point-like angular momentum distribution (spin) and no spin-orbit coupling, a non-symmetrical stress tensor is not acceptable. Decades later, Landau and Lifshits proposed a symmetrical gravitational energy-momentum $t^{\lambda\mu}(x)$, satisfying the following equation[3]

$$\frac{\partial}{\partial x^\lambda}[-|g(x)|(T^{\lambda\mu}(x)+t^{\lambda\mu}(x))]=0, \forall\mu=0,1,2,3$$

which is equivalent to

$$\int_{\partial\Omega} ds_\lambda(x)(-|g(x)|)(T^{\lambda\mu}(x)+t^{\lambda\mu}(x))=0.\forall\mu=0,1,2,3 \tag{8}$$

However, its volume element $d^4x(-|g(x)|)$ and hyper-surface element $ds_\lambda(x)(-|g(x)|)$ do not have the correct transformation property under general coordinate transformations. Hence $ds_\lambda(x)(-|g(x)|)T^{\lambda\mu}(x)$ can not be taken as the $\mu$-component of matter energy-momentum on small hyper-surface element at $x$, let alone $\int_\Sigma ds_\lambda(x)(-|g(x)|)T^{\lambda\mu}(x)$ be taken as the $\mu$-component of matter energy-momentum 4-vector on hyper-surface $\Sigma$.

Following Einstein, Tolman, Landau, Lifshits, and Møller et al. proposed several gravitational energy-momentum complexes[4],[5]. They are all pseudotensors in the following sense.

$$t^{\lambda\mu}(y)\neq\frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial y^\mu}{\partial x^\beta}t^{\alpha\beta}(x), \forall 0\leqslant\lambda,\mu\leqslant 3 \tag{9}$$

One of the direct consequences of pseudotensor character is the non-localizability. Efforts to search for a covariant localizable description of gravitational energy-momentum have never ceased; but all of them failed. Pseudotensor character and non-localizability of gravitational energy-momentum are attributed to the equivalence principle physically, and to the following fact mathematically: For any geodesic $\gamma$ in spacetime, one can always choose coordinates $\{x\}$, such that all the Chritoffel symbols $\Gamma^\alpha_{\beta\gamma}(x)$ vanish at all $p$ on $\gamma$ (See, e.g.,[6]). Accepting that the non-localizability of gravitational energy-momentum is an unavoidable consequence of equivalence principle, some relativists switched to search for the total gravitational energy when spacetime is asymptotically flat at spacelike and null infinity[7],[8],[9]. The proof of the positivity of the ADM mass and Bondi mass is considered one of the greatest achievements in classical GR in the last 35 years[10],[11]. This success inspired the search for quasi-local conserved quantities. But, finding an appropriate quasi-local notion of energy-momentum has proven to be surprisingly difficult (See [12]).

The propagating solutions of Einstein's field equation clearly exist. What is at issue is whether they carry energy-momentum or not. This issue dates back to A.C. Eddington. Even Einstein himself has said, "There may very well be gravitational fields without stress and energy density" in his response to Schrödinger's comment. R.P. Feynman's "rod plus sticky beads" detector presented at 1957 Chapel Hill conference convinced most relativists of gravitational energy radiation[13]. Enormous efforts to detect gravitational waves have been made. But most of them failed, and Hulse and Taylor's discovery of a new type of pulsar is considered an indirect evidence of gravitational radiation[14]. 100 years after GR was founded by Einstein, the detecting of gravitational wave finally was reported by LIGO[15]. This experiment measures change of spacetime geometry unlike Feynman's detector which detectes energy brought by gravitational wave. Can LIGO's result be taken as an evidence for gravitational energy radiation?

The aim of the present paper is to show:

(i) The long existing pseudotensor, non-localizability problem of gravitational energy-momentum is a result of mistaking different geometrical, physical objects as one and the same.

(ii) In curved spacetime, the sum 4-vector of matter energy-momentum on a finite or an infinite hyper-surface does not have any meaning, but the density-flux tensor field $T$ of matter energy-momentum 4-vector $P$ does (See Section 2). In curvilinear coordinate systems conservation of matter energy-momentum is not eqn.(1), the conservation of its components. While eqn.(3), the vanishing of covariant divegence of matter stress tensor field, means "there is no spring or sink for matter energy-momentum anywhere in spacetime". It is the proper expression for matter energy-momentum conservation, contrary to the commonly accepted viewpoint. This expression for matter energy-momentum conservation is independent of coordinates and good for all kinds of spacetimes, flat or curved, with a pre-given metric or with a pending metric to be determined by the least action principle.

(iii) After carefully defining "the change of a particle's energy-momentum" (See section 2), we show in the case of classical electrodynamics, a free particle's energy-momentum does not change, while the change of a charged particle's energy-momentum during $d\tau$ ($\tau$ is the proper time) is exactly the amount that the electromagnetic field gives it during $d\tau$. Now that, there is no spring or sink everywhere in spactime for matter (particles' plus electromagnetic field's) energy-momentum, therefore gravitational field does not exchange energy-momentum with both electromagnetic field and particles (charged and uncharged). Hence it does not carry energy-momentum. Gravitational field is not a force field, and gravity is not a natural force.

The whole argument of the present paper will be based on a general variational principle, Einstein-Hilbert action and sound geometrical facts.

## 2 Some facts from geometry

Some geometrical, physical concepts formed from our experiences in flat spacetime, such as free vectors, displacement vectors, sum of distributed vectors over a hyper-surface, etc. no longer make sense in curved spacetime. Picture thinking often leads to misunderstanding in GR.

(1) In special relativity (SR), we can talk about the sum of two particles' energy-momentum 4-vectors on a given spacelike hyper-surface. But in GR, we can not talk about it. Suppose the two particles' world-lines (with proper times as their parameters) intersect the spacelike hyper-surface $\Sigma$ at $A$, $B$ respectively. A particle's energy-momentum vector is its rest mass times the tangent vector to its world-line. The two particles' energy-momentum vectors $P_A$ and $P_B$ belong to different tangent spaces $T_A$ and $T_B$ respectively. No one can add up vectors from different vector spaces. In order to add them up, one has to parallelly transport them to the same spacetime point, say $C$. Then, in the same tangent space $T_C$ one can add up the parallelly transported vectors $P'_{AC}$ and $P'_{BC}$. If the spacetime is flat where parallel transport is independent of the path, this is OK, and we have the concept of free vectors. But in a curved spacetime, parallel transport depends on the path. In order to get rid of the ambiguity, one might suggest parallelly transporting $P_A$ and $P_B$ along geodesics to $C$. But the sum vector $P'_{AC} + P'_{BC} \in T_C$ still depends on $C$ in the following sense: If one chooses spacetime point $D$ instead of $C$, then parallelly transporting vector $P_A$ and $P_B$ along the geodesics to $D$, one gets $P'_{AD} + P'_{BD} \in T_D$. Parallelly transporting $P'_{AC} + P'_{BC}$ along the geodesic from $C$ to $D$ does not in general result in $P'_{AD} + P'_{BD}$. Therefore we can not define a sum vector of $P_A$ and $P_B$ independent of individual's subjective will. In general, in a curved spacetime we can not add up an $(r, s)$-tensor, point-likely or continuously distributed over different points of spacetime, unless $r = s = 0$.

(2) In SR, the spacetime is Minkowski space, where we have the concept of the finite displacement 4-vector $\boldsymbol{\rho}(\tau_1, \tau_2)$ of a particle first, and then define the velocity 4-vector $\mathbf{v}$ of the particle by using it as follows.

$$\lim_{\tau_2 \to \tau_1} \frac{\boldsymbol{\rho}(\tau_1, \tau_2)}{\tau_2 \ - \tau_1} =: \mathbf{v}(\tau_1) \tag{10}$$

However, in GR the spacetime $M$ is a generalized Riemannian manifold with a pending Lorentzian metric field (to be determined by the least action principle), and the difference of a particle's coordinates at proper times $\tau_1$ and $\tau_2$, $(\triangle x^0, \triangle x^1, \triangle x^2, \triangle x^3)$, does not transform like a vector under arbitrary coordinate transformations. So we do not have the concept of the displacement 4-vector of a particle. However,

$$(\lim_{\tau_2 \to \tau_1} \frac{\triangle x^0}{\triangle \tau}, \lim_{\tau_2 \to \tau_1} \frac{\triangle x^1}{\triangle \tau}, \lim_{\tau_2 \to \tau_1} \frac{\triangle x^2}{\triangle \tau}, \lim_{\tau_2 \to \tau_1} \frac{\triangle x^3}{\triangle \tau})$$

transforms like a vector under arbitrary coordinate transformations, hence we can define velocity 4-vector of a particle in curved spacetime despite failure to define its finite displacement 4-vector. Remembering how the tangent vectors at a point of a differential manifold are defined in modern geometry, one would agree that in GR the matter energy-momentum density-flux tensor is well defined, even though the sum 4-vector of matter energy-momentum over a finite hyper-surface can not be defined. In general, in curved spacetime the density-flux $(1+r,s)$-tensor field of an $(r,s)$-tensor $T$ can be defined, while the sum tensor of distributed $T$ over a finite hyper-surface can't when $r+s>0$.

(3) In GR, even the concept of the change of one particle's energy-momentum 4-vector has to be carefully defined. Suppose the particle's world line is $\gamma:\Delta\longrightarrow M$, where $\Delta=:[\tau_i,\tau_f]$, $\tau_i$ ($\tau_f$) is the proper time when the particle is created (annihilated), or $\Delta=:[\tau_i,+\infty)$, $(-\infty,\tau_f]$, $(-\infty,+\infty)$. Its energy-momentum 4-vectors at proper times $\tau_1,\tau_2\in\Delta$ ($\tau_1\neq\tau_2$), belong to different tangent spaces. We can not subtract one from the other. Denote by $P(\tau)$ ($\in T_{\gamma(\tau)}$) the particle's energy-momentum 4-vector at proper time $\tau\in\Delta$, and for $\tau_0,\tau\in\Delta$, denote by $\widetilde{P}_{\tau_0}(\tau)$ the vector obtained by parallelly transporting $P(\tau_0)$ along the world line from $\gamma(\tau_0)$ to $\gamma(\tau)$. The reasonable definition of the change of a particle's energy-momentum 4-vector during proper time interval $[\tau_1,\tau_2]\subset\Delta$ is a vector field defined only on its world line

$$\delta_{\tau_1,\tau_2}P:\Delta\longrightarrow\bigcup_{\tau\in\Delta}T_{\gamma(\tau)} \tag{11}$$

such that

$$\delta_{\tau_1,\tau_2}P(\tau)=:\widetilde{P}_{\tau_2}(\tau)-\widetilde{P}_{\tau_1}(\tau)\in T_{\gamma(\tau)} \tag{12}$$

It is easy to check,

$$\delta_{\tau_1,\tau_2}P+\delta_{\tau_2,\tau_3}P=\delta_{\tau_1,\tau_3}P \tag{13}$$

So, this definition is meaningful. Should we define the change of a particle's energy-momentum 4-vector during proper time interval $[\tau_1,\tau_2]\in\Lambda$ as $\delta_{\tau_1,\tau_2}P(\tau)=:\overline{P}_{\tau_2}(\tau)-\overline{P}_{\tau_1}(\tau)\in T_{\gamma(\tau)}$, where $\overline{P}_{\tau_2}(\tau)$ is the vector obtained by parallelly transporting $P(\tau_2)$ along the geodesic from $\gamma(\tau_2)$ to $\gamma(\tau)$, the above self-consistency (13) would fail. Therefore, if we wish to talk about the change of a particle's energy-momentum 4-vector in curved spacetime, (11)+(12) is the only reasonable definition. It is worth noting, this definition does not depend on coordinates.

(4) Now we are in a position to explore the meaning of conservation laws in GR.

(i) For a scalar $S$, let $J$ be the density-flux vector field of $S$. As pointed above, we can add up a distributed scalar over a hyper-surface, no matter the spacetime is flat or curved. In any coordinate system $\{x\}$, integral $\int_\Sigma ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x)$ is the sum of scalar $S$ distributed on spacelike hyper-surface $\Sigma$, and integral $\int_\Gamma ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x)$ is the amount of scalar $S$ flowing through timelike hyper-surface $\Gamma$. Let $\Omega$ be an arbitrary spacetime region with boundary $\partial\Omega$ composed of a past spacelike hyper-surface $\Sigma$, a future spacelike hyper-surface $\Sigma'$ and a timelike hyper-surface $\Gamma$ which links the boundaries of $\Sigma$ and $\Sigma'$. We have

$$\int_\Omega d^4x\sqrt{-|g(x)|}\nabla_\lambda J^\lambda(x)=\int_\Omega d^4x\frac{\partial}{\partial x^\lambda}\left(\sqrt{-|g(x)|}J^\lambda(x)\right)$$

$$= \int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x) = \int_{\Sigma'} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x)$$

$$- \int_{\Sigma} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x) + \int_{\Gamma} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x) \tag{14}$$

The right hand side of eqn.(14) is the amount of scalar $S$ distributed on $\Sigma'$ plus the amount of scalar $S$ flowing away though $\Gamma$ minus the amount of scalar $S$ distributed on $\Sigma$. Hence the left hand side is the amount of scalar $S$ created in $\Omega$. And considering that $d^4x\sqrt{-|g(x)|}$ is the invariant 4-volume element, we conclude that the covariant divergence of $J$, $\nabla_\lambda J^\lambda(x)$, is the amount of scalar $S$ created in per unit 4-volume. Therefore, the law of conservation of a scalar $S$, should be the covariant divergence of its density-flux vector field, $\nabla_\lambda J^\lambda(x)$, vanishes everywhere in spacetime $M$:

$$\nabla_\lambda J^\lambda(x)|_p = 0, \forall p \in M \tag{15}$$

It is equivalent to the continuity equation

$$\int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda(x) = 0, \forall \text{ spacetime region } \Omega \tag{16}$$

(ii) We will confine ourselves to flat spacetime in this paragraph, so that we can add up an $(r,s)$-tensor $A$ distributed over a finite hyper-surface or created in a finite spacetime region. Denote by $B$ the density-flux $(1+r,s)$-tensor field of $A$. In a flat spacetime, there exist flat coordinate systems, in particular, there exist inertial coordinate systems. In an inertial coordinate system $\{y\}$ ($g_{\mu\nu}(y) \equiv \eta_{\mu\nu} = diag\{-1,1,1,1\}$), for any spacetime region $\Omega$, we have, for $0 \leqslant \mu_1, \cdots, \mu_r, \nu_1, \cdots, \nu_s \leqslant 3$

$$\int_{\Omega} d^4y\sqrt{-|g(y)|}\nabla_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y) = \int_{\Omega} d^4y\partial_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y)$$

$$= \int_{\partial\Omega} ds_\lambda(y)B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y) = \left(\int_{\Sigma'} - \int_{\Sigma} + \int_{\Gamma}\right) ds_\lambda(y)B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y)$$

The RHS is the components in inertial coordinate system $\{y\}$ of the amount of $A$ distributed on $\Sigma'$ minus the amount of $A$ distributed on $\Sigma$ plus the amount of $A$ flowing away through $\Gamma$, $(A|_{\Sigma'} - A|_{\Sigma} + A|_{\Gamma})^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y)$. Hence the LHS is the components in inertial coordinate system $\{y\}$ of the amount of $A$ created in $\Omega$. Considering $d^4y\sqrt{-|g(y)|}$ is the invariant 4-volume element, $\nabla_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y)$ should be the components in inertial coordinate system $\{y\}$ of the amount of A created in per unit invariant 4-volume. The law of conservation of $(r,s)$-tensor $A$ in flat spacetime can be expressed in an inertial coordinate system $\{y\}$ as $4^{r+s}$ continuity equations

$$\int_{\partial\Omega} ds_\lambda(y)B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y) = 0, \forall 0 \leqslant \mu_i, \nu_j \leqslant 3 \tag{17}$$

or equivalently

$$\partial_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y)=\nabla_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(y)=0 \tag{18}$$

In an arbitrary coordinate system $\{x\}$, the above equations are equivalent to

$$\nabla_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(x)=0,\forall 0\leqslant\mu_1,\cdots,\mu_r,\nu_1,\cdots,\nu_s\leqslant 3 \tag{19}$$

or

$$\int_{\partial\Omega}ds_\lambda(x)\sqrt{-|g(x)|}B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(x)=\int_\Omega d^4x\sqrt{-|g(x)|}\sum_{1\leqslant j\leqslant s}\Gamma^\sigma_{\lambda\nu_j}B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\sigma\cdots\nu_s}(x)$$
$$-\int_\Omega d^4x\sqrt{-|g(x)|}\sum_{1\leqslant i\leqslant r}\Gamma^{\mu_i}_{\lambda\rho}B^{\lambda\mu_1\cdots\rho\cdots\mu_r}_{\nu_1\cdots\nu_s}(x),\forall 0\leqslant\mu_1,\cdots,\mu_r,\nu_1,\cdots,\nu_s\leqslant 3 \tag{20}$$

No one denies that eqns.(17) and (18) are the proper expressions for conservation of $(r,s)$-tensor $A$. Now that eqns.(19) and (20) are equivalent to eqns.(17) and (18), so they are the proper expressions for conservation of $(r,s)$-tensor $A$ too. Some leading scholars (say, Bondi[16], Nester[17]) think the non-vanishing RHS of eqn.(20) ruins the conservation of $(r,s)$-tensor $A$. Regarding this, we just point out that, in a curvilinear coordinate system, for a spacelike hyper-surface $\Sigma$, $\int_\Sigma ds_\lambda(x)\sqrt{-|g(x)|}B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(x)$ is not the $\binom{\mu_1\cdots\mu_r}{\nu_1\cdots\nu_s}$-components of sum $(r,s)$-tensor $A$ on $\Sigma$; for a timelike hyper-surface $\Gamma$, $\int_\Gamma ds_\lambda(x)\sqrt{-|g(x)|}B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(x)$ is not the $\binom{\mu_1\cdots\mu_r}{\nu_1\cdots\nu_s}$-components of sum $(r,s)$-tensor $A$ flowing though $\Gamma$. Conservation of $(r,s)$-tensor $A$ (expressed by eqns.(17) through (20)) is not conservation of its components in a curvilinear coordinate system $\{x\}$

$$\int_{\partial\Omega}ds_\lambda(x)\sqrt{-|g(x)|}B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(x)\ =0,\forall 0\leq\mu_i,\nu_j\leq 3,\Omega\subset M.$$

(iii) Let's generalize the above to the case of conservation law for an $(r,s)$-tensor in curved spacetime. Note that the conservation law is an objective truth, it should not depend on the coordinate systems chosen by individuals. And note that the general conservation law should give the well-established results: eqn.(15) (or eqn.(16)), when $r=s=0$; eqns.(17) through (20), when the spacetime is flat. Now that in curved spacetime, the sum vector of matter energy-momentum over a finite hypersurface or created in a finite spacetime region no longer make sense, and we can only talk about matter energy-momentum over an infinitesimal hypersurface or created in an infinitesimal spacetime region, hence the only possibility is

**Proposition 1** *The proper expression for conservation law of an $(r,s)$-tensor $A$ is the covariant divergence of its density-flux $(1+r,s)$-tensor field $B$ vanishes everywhere in spacetime $M$:*

$$\nabla_\lambda B^{\lambda\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s}(x)|_p=0,\forall 0\leq\mu_i,\nu_j\leq 3\ \&\ p\in M \tag{21}$$

*or any differential (integral) equation equivalent to it.*

In particular, eqn.(3) itself is the law of conservation of matter energy-momentum in GR. Introducing the gravitational energy-momentum $\tau^{\lambda\mu}(x)$ to save the law of conservation of energy-momentum in GR is improper.

(5) Density-flux for point-like distributed quantity

The covariant electrical density-flux 4-vector field of charged particles is

$$j^{\lambda}(x) = \sum \int_{\Delta} d\tau \left[ \frac{dx^{\lambda}(\gamma(\tau))}{d\tau} \frac{\delta^4 \left( x - x(\gamma(\tau)) \right)}{\sqrt{-|g(x)|}} cq \right] \tag{22}$$

where $q$ is the particle's charge, $\gamma : \Delta \longrightarrow M$ is the particle's world line, (see paragraph (3) above) $\tau$ is the proper time and summation is taken over all particles.

$$\begin{aligned}
\int_{\Omega} d^4x \sqrt{-|g(x)|} \nabla_{\lambda} j^{\lambda}(x) &= \int_{\Omega} d^4x \frac{\partial}{\partial x^{\lambda}} [\sqrt{-|g(x)|} j^{\lambda}(x)] \\
&= \int_{\Omega} d^4x \sum \int_{\Delta} d\tau \; cq \; \frac{\partial}{\partial x^{\lambda}} \delta^4 \left( x - x(\gamma(\tau)) \right) \frac{dx^{\lambda}(\gamma(\tau))}{d\tau} \\
&= - \int_{\Omega} d^4x \sum \int_{\Delta} d\tau \; cq \; \frac{\partial}{\partial x^{\lambda}(\gamma(\tau))} \delta^4 \left( x - x(\gamma(\tau)) \right) \frac{dx^{\lambda}(\gamma(\tau))}{d\tau} \\
&= - \int_{\Omega} d^4x \sum \int_{\Delta} d\tau \; cq \; \frac{d}{d\tau} \delta^4 \left( x - x(\gamma(\tau)) \right) \\
&= - \sum \int_{\Delta} d\tau \; cq \; \frac{d}{d\tau} H_{\Omega}(\gamma(\tau))
\end{aligned} \tag{23}$$

where $H_{\Omega}$ is the Heaviside function defined on spacetime $M$,

$$\begin{aligned}
H_{\Omega}(p) &= 1, \text{ if } p \in \Omega \\
H_{\Omega}(p) &= 0, \text{ if } p \in M \backslash \Omega
\end{aligned} \tag{24}$$

Then we get

$$\oint_{\partial\Omega} ds_{\alpha}(x) \sqrt{-|g(x)|} j^{\alpha}(x) = - \sum \int_{\gamma} cq \; dH_{\Omega}(\gamma(\tau)) \tag{25}$$

It equals $c$ times the charges created in $\Omega$ minus the charges annihilated in $\Omega$. When for all particles $\Delta = (-\infty, +\infty)$, we have charge conservation.

The above observations are sound facts from geometry, and their validity does not rely on the equivalence principle.

# 3 Variational principle for Mach and non-Mach dynamics

All the dynamics other than GR study how the state of matter evolves in spacetime with a metric field given before solving the equations of motion. While

GR studies how the state of matter evolves in spacetime and conversely, how the matter movement determines the spacetime metric. In the latter case, the spacetime metric is not given in advance, it is determined along with the matter movement at the same time by solving the equations of motion (by the least action principle). In this context, in GR both the spacetime metric field and the variable describing matter movement are called dynamic variables (The way it is called doesn't change spacetime metric's pure geometrical characteristic). Dynamics with a pre-given spacetime metric will be called non-Mach dynamics, no matter the pre-given spacetime background is flat or curved (say, it can be the Minkowski space or Schwarzschild spacetime), and GR is the only Mach dynamics. The variational principles and Noether's theorems in GR and in non-Mach dynamics are significantly different.

We assume here the matter field $u$ is a $(1,1)$-type tensor field. But the result can be readily generalized to cases of any $(r,s)$-tensor matter field. The Einstein-Hilbert action of the dynamic system over spacetime region $\Omega$

$$\mathcal{A}[\Omega; g, u] = \mathcal{A}_M[\Omega; g, u] + \mathcal{A}_G[\Omega; g]$$

$$=: \int_\Omega d^4x\sqrt{-|g(x)|}[\mathcal{L}(g(x), u(x), \nabla u(x)) + \frac{c^3 R(g(x), \partial g(x), \partial^2 g(x))}{16\pi G}] \quad (26)$$

will be used as the starting point, where $\mathcal{L}$ is the sum of a few scalars obtained by contracting $g, u$ and $\nabla u$, and multiplying the contractions by proper coefficients, such that $\mathcal{L}(\eta, u(x), \partial u(x))$ $(\eta = diag(-1,1,1,1))$ is the Lagrangian in special relativity, and $R$ is Ricci's scalar curvature.

## 3.1 Equations of motion in GR

In GR, the difference of actions over spacetime region $\Omega$ of two kinematically allowed movements close to each other, $(u, g)$ and $(\widetilde{u} = u + \overline{\delta} u, \widetilde{g} = g + \overline{\delta} g)$, is

$$\mathcal{A}_M[\Omega; \widetilde{g}, \widetilde{u}] - \mathcal{A}_M[\Omega; g, u] = \int_\Omega d^4x\{\mathcal{L}(g(x), u(x), \nabla u(x))\overline{\delta}\sqrt{-|g(x)|}$$

$$+\sqrt{-|g(x)|}[\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}(x)}\overline{\delta} g_{\alpha\beta}(x) + \frac{\partial \mathcal{L}}{\partial u^\theta_\xi(x)}\overline{\delta} u^\theta_\xi(x) + \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\overline{\delta}\nabla_\lambda u^\theta_\xi(x)]\} \quad (27)$$

Because

$$\overline{\delta}\nabla_\lambda u^\theta_\xi(x) = \nabla_\lambda \overline{\delta} u^\theta_\xi(x) + u^\varphi_\xi(x)\overline{\delta}\Gamma^\theta_{\lambda\varphi}(x) - u^\theta_\eta(x)\overline{\delta}\Gamma^\eta_{\lambda\xi}(x)$$

and

$$\sqrt{-|g(x)|}\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\nabla_\lambda \overline{\delta} u^\theta_\xi(x)$$

$$= \sqrt{-|g(x)|}\nabla_\lambda[\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\overline{\delta} u^\theta_\xi(x)] - \sqrt{-|g(x)|}[\nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}]\overline{\delta} u^\theta_\xi(x)$$

$$= \frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\overline{\delta} u^\theta_\xi(x)] - \sqrt{-|g(x)|}[\nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}]\overline{\delta} u^\theta_\xi(x)$$

we have

$$\mathcal{A}_M[\Omega;\widetilde{g},\widetilde{u}]-\mathcal{A}_M[\Omega;g,u]=\int_\Omega d^4x\sqrt{-|g(x)|}\{[(\frac{1}{2}g^{\alpha\beta}(x)\mathcal{L}$$

$$+\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}(x)})\overline{\delta}g_{\alpha\beta}(x)]+[(\frac{\partial\mathcal{L}}{\partial u^\theta_\xi(x)}-\nabla_\lambda\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)})\overline{\delta}u^\theta_\xi(x)]$$

$$+[\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}(u^\varphi_\xi(x)\overline{\delta}\Gamma^\theta_{\lambda\varphi}(x)-u^\theta_\eta(x)\overline{\delta}\Gamma^\eta_{\lambda\xi}(x))]\}$$

$$+\int_\Omega d^4x\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}\overline{\delta}u^\theta_\xi(x)\} \qquad (28)$$

It is worth noting that the terms in the three square brackets at right hand side are all scalars. Because

$$\int_\Omega d^4x\sqrt{-|g(x)|}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}u^\varphi_\xi(x)\overline{\delta}\Gamma^\theta_{\lambda\varphi}(x)$$

$$=\int_\Omega d^4x\sqrt{-|g(x)|}\nabla_\lambda[-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}u^\beta_\xi(x)g^{\theta\alpha}(x)$$

$$-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\xi(x)}u^\lambda_\xi(x)g^{\theta\alpha}(x)+\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\xi(x)}u^\beta_\xi(x)g^{\theta\lambda}(x)]\overline{\delta}g_{\alpha\beta}(x)$$

$$+\int_\Omega d^4x\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}u^\beta_\xi(x)g^{\theta\alpha}(x)$$

$$+\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\xi(x)}u^\lambda_\xi(x)g^{\theta\alpha}(x)-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\xi(x)}u^\beta_\xi(x)g^{\theta\lambda}(x)]\overline{\delta}g_{\alpha\beta}(x)\}, \qquad (29)$$

and

$$-\int_\Omega d^4x[\sqrt{-|g(x)|}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}u^\theta_\eta(x)\overline{\delta}\Gamma^\eta_{\lambda\xi}(x)]$$

$$=\int_\Omega d^4x\sqrt{-|g(x)|}\nabla_\lambda[\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\beta(x)}u^\theta_\eta(x)g^{\eta\alpha}(x)$$

$$+\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\lambda(x)}u^\theta_\eta(x)g^{\eta\alpha}(x)-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\beta(x)}u^\theta_\eta(x)g^{\eta\lambda}(x)]\delta g_{\alpha\beta}(x)$$

$$+\int_\Omega d^4x\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\beta(x)}u^\theta_\eta(x)g^{\eta\alpha}(x)$$

$$-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\lambda(x)}u^\theta_\eta(x)g^{\eta\alpha}(x)+\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\beta(x)}u^\theta_\eta(x)g^{\eta\lambda}(x)]\overline{\delta}g_{\alpha\beta}(x)\}, \qquad (30)$$

we have

$$\mathcal{A}_M[\Omega;\widetilde{g},\widetilde{u}]-\mathcal{A}_M[\Omega;g,u]=\int_\Omega d^4x\sqrt{-|g(x)|}\{\frac{1}{2}T^{\alpha\beta}(x)\overline{\delta}g_{\alpha\beta}(x)$$

$$+[\frac{\partial\mathcal{L}}{\partial u^\theta_\xi(x)}-\nabla_\lambda\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}]\overline{\delta}u^\theta_\xi(x)\}$$

$$+\int_\Omega d^4x\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}\overline{\delta}u^\theta_\xi(x)+U^{\lambda\alpha\beta}(x)\overline{\delta}g_{\alpha\beta}(x)]\},\qquad(31)$$

where

$$U^{\lambda\alpha\beta}(x)=U^{\lambda\beta\alpha}(x)=:$$

$$\frac{1}{4}[\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}u^\beta_\xi(x)g^{\theta\alpha}(x)+\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\xi(x)}u^\lambda_\xi(x)g^{\theta\alpha}(x)-\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\xi(x)}u^\beta_\xi(x)g^{\theta\lambda}(x)$$

$$-\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\beta(x)}u^\theta_\eta(x)g^{\eta\alpha}(x)-\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\lambda(x)}u^\theta_\eta(x)g^{\eta\alpha}(x)+\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\beta(x)}u^\theta_\eta(x)g^{\eta\lambda}(x)$$

$$+\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}u^\alpha_\xi(x)g^{\theta\beta}(x)+\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\xi(x)}u^\lambda_\xi(x)g^{\theta\beta}(x)-\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\xi(x)}u^\alpha_\xi(x)g^{\theta\lambda}(x)$$

$$-\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\alpha(x)}u^\theta_\eta(x)g^{\eta\beta}(x)-\frac{\partial\mathcal{L}}{\partial\nabla_\alpha u^\theta_\lambda(x)}u^\theta_\eta(x)g^{\eta\beta}(x)+\frac{\partial\mathcal{L}}{\partial\nabla_\beta u^\theta_\alpha(x)}u^\theta_\eta(x)g^{\eta\lambda}(x)]\qquad(32)$$

is a $(3,0)$-tensor field symmetrical with respect to indices $\alpha$ and $\beta$, and the energy-momentum tensor of matter

$$T^{\alpha\beta}(x)=:\frac{2}{\sqrt{-|g(x)|}}\frac{\delta\mathcal{A}_M}{\delta g_{\alpha\beta}(x)}=2[\frac{1}{2}g^{\alpha\beta}(x)\mathcal{L}+\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}(x)}-\nabla_\lambda U^{\lambda\alpha\beta}(x)]\qquad(33)$$

is a symmetrical $(2,0)$-tensor field. While

$$\mathcal{A}_G[\Omega;\widetilde{g}]-\mathcal{A}_G[\Omega;g]=\int_\Omega d^4x\sqrt{-|g(x)|}\frac{c^3}{16\pi G}(\frac{1}{2}Rg^{\alpha\beta}(x)$$

$$-R^{\alpha\beta})\overline{\delta}g_{\alpha\beta}(x)+\int_\Omega d^4x[\sqrt{-|g(x)|}\frac{c^3}{16\pi G}g^{\mu\nu}(x)\delta R_{\mu\nu}(x)]$$

Because

$$\begin{aligned}\overline{\delta}R_{\mu\nu}(x)&=\nabla_\lambda\overline{\delta}\Gamma^\lambda_{\mu\nu}(x)-\nabla_\mu\overline{\delta}\Gamma^\lambda_{\lambda\nu}(x),\\ \sqrt{-|g(x)|}g^{\mu\nu}(x)\overline{\delta}R_{\mu\nu}(x)&=\sqrt{-|g(x)|}\nabla_\lambda[g^{\mu\nu}(x)\overline{\delta}\Gamma^\lambda_{\mu\nu}(x)-g^{\lambda\nu}(x)\overline{\delta}\Gamma^\mu_{\mu\nu}(x)]\\ &=\frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}[g^{\mu\nu}(x)\overline{\delta}\Gamma^\lambda_{\mu\nu}(x)-g^{\lambda\nu}(x)\overline{\delta}\Gamma^\mu_{\mu\nu}(x)]\end{aligned}$$

Therefore

$$\mathcal{A}_G[\Omega;\widetilde{g}]-\mathcal{A}_G[\Omega;g] = \int_\Omega d^4x\sqrt{-|g(x)|}\frac{c^3}{16\pi G}(\frac{1}{2}Rg^{\alpha\beta}(x)-R^{\alpha\beta})\overline{\delta}g_{\alpha\beta}(x) + \int_\Omega d^4x\frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}\frac{c^3}{16\pi G}W^\lambda(x)] \tag{34}$$

where

$$W^\lambda(x) =: g^{\alpha\beta}(x)\overline{\delta}\Gamma^\lambda_{\alpha\beta}(x)-g^{\alpha\lambda}(x)\overline{\delta}\Gamma^\beta_{\beta\alpha}(x)$$

$$=\frac{1}{2}[g^{\lambda\tau}(x)g^{\alpha\rho}(x)g^{\beta\sigma}(x)+\frac{1}{2}g^{\lambda\alpha}(x)g^{\beta\tau}(x)g^{\rho\sigma}(x)+\frac{1}{2}g^{\lambda\beta}(x)g^{\alpha\tau}(x)g^{\rho\sigma}(x)$$
$$-g^{\lambda\alpha}(x)g^{\beta\rho}(x)g^{\tau\sigma}(x)-g^{\lambda\beta}(x)g^{\alpha\rho}(x)g^{\tau\sigma}(x)]\partial_\tau g_{\rho\sigma}(x)\overline{\delta}g_{\alpha\beta}(x)$$
$$+\frac{1}{2}[g^{\lambda\alpha}(x)g^{\gamma\beta}(x)+g^{\lambda\beta}(x)g^{\gamma\alpha}(x)-2g^{\lambda\gamma}(x)g^{\alpha\beta}(x)]\overline{\delta}\partial_\gamma g_{\alpha\beta}(x)$$
$$=: E^{\lambda\alpha\beta}(x)\overline{\delta}g_{\alpha\beta}(x)+F^{\lambda\gamma\alpha\beta}(x)\overline{\delta}\partial_\gamma g_{\alpha\beta}(x) \tag{35}$$

is a tangent field ($E^{\lambda\alpha\beta}(x)\overline{\delta}g_{\alpha\beta}(x)$ and $F^{\lambda\gamma\alpha\beta}(x)\overline{\delta}\partial_\gamma g_{\alpha\beta}(x)$ are not tangent fields individually). We get

$$\mathcal{A}[\Omega;\widetilde{g},\widetilde{u}]-\mathcal{A}[\Omega;g,u]=\int_\Omega d^4x\sqrt{-|g(x)|}\{[\frac{\partial\mathcal{L}}{\partial u^\theta_\xi(x)}-\nabla_\lambda\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}]\overline{\delta}u^\theta_\xi(x)$$
$$+\frac{c^3}{16\pi G}[\frac{8\pi G}{c^3}T^{\alpha\beta}(x)-(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}(x))]\overline{\delta}g_{\alpha\beta}(x)\}$$
$$+\int_{\partial\Omega} ds_\lambda(x)\{\sqrt{-|g(x)|}[\frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}\delta u^\theta_\xi(x)+(U^{\lambda\alpha\beta}(x)+$$
$$\frac{c^3}{16\pi G}E^{\lambda\alpha\beta}(x)\overline{\delta}g_{\alpha\beta}(x)+\frac{c^3}{16\pi G}\ F^{\lambda\gamma\alpha\beta}(x)\overline{\delta}\partial_\gamma g_{\alpha\beta}(x)]\} \tag{36}$$

By using **the least action principle**: *For any spacetime region $\Omega$, among all kinematically allowed movements in $\Omega$ with the same boundary value*

$$\overline{\delta}u|_{\partial\Omega}=0,\overline{\delta}g|_{\partial\Omega}=0,\overline{\delta}\partial g|_{\partial\Omega}=0 \tag{37}$$

*the movement allowed by physical laws takes the stationary value of the action over $\Omega$,* we get the equations of motion

$$\frac{\partial\mathcal{L}(g(x),u(x),\nabla u(x))}{\partial u^\theta_\xi(x)}-\nabla_\lambda\frac{\partial\mathcal{L}(g(x),u(x),\nabla u(x))}{\partial\nabla_\lambda u^\theta_\xi(x)}=0 \tag{38}$$

$$R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta}(x)=\frac{8\pi G}{c^3}T^{\alpha\beta}(x) \tag{39}$$

### 3.2 Equation of motion in non-Mach dynamics

For the non-Mach dynamics of the $(1,1)$-type tensor matter field $u$, we use the same action (26). But the spacetime metric field $g$ here, is given in advance, and is no longer a dynamic variable determined by the least action principle (It is called a non-variational field in [18]). The difference of actions over spacetime region $\Omega$ of two kinematically allowed movements close to each other, $u$ and $\widetilde{u} = u + \overline{\delta}u$, is

$$\begin{aligned}\mathcal{A}[\Omega; g, \widetilde{u}] - \mathcal{A}[\Omega; g, u] &= A_M[\Omega; g, \widetilde{u}] - A_M[\Omega; g, u] \\ = \int_\Omega d^4x\sqrt{-|g(x)|}[\frac{\partial \mathcal{L}}{\partial u^\theta_\xi(x)}\overline{\delta}u^\theta_\xi(x) &+ \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\overline{\delta}\nabla_\lambda u^\theta_\xi(x)] \\ = \int_\Omega d^4x\sqrt{-|g(x)|}[\frac{\partial \mathcal{L}}{\partial u^\theta_\xi(x)} &- \nabla_\lambda\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}]\overline{\delta}u^\theta_\xi(x) \\ + \int_{\partial\Omega} ds_\lambda(x)[\sqrt{-|g(x)|}&\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\overline{\delta}u^\theta_\xi(x)]\end{aligned} \tag{40}$$

The Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}(g(x), u(x), \nabla u(x))}{\partial u^\theta_\xi(x)} - \nabla_\lambda \frac{\partial \mathcal{L}(g(x), u(x), \nabla u(x))}{\partial \nabla_\lambda u^\theta_\xi(x)} = 0. \tag{41}$$

(Note that Einstein's field equation (39) is no longer equation of motion for non-Mach dynamics) This equation of motion holds in all coordinate systems, no matter what the pre-given spacetime background is. It is covariant under general coordinate transformations. In the specific non-Mach dynamics, SR, the pre-given spacetime background is Minkowski space. There exist inertial coordinate systems. In an inertial coordinate system $\{y^0, y^1, y^2, y^3\}$, eqn.(41) can be written as

$$\frac{\partial \mathcal{L}(\eta, u(y), \partial u(y))}{\partial u^\theta_\xi(y)} - \partial_\lambda \frac{\partial \mathcal{L}(\eta, u(y), \partial u(y))}{\partial \partial_\lambda u^\theta_\xi(y)} = 0. \tag{42}$$

# 4 Noether's theorems for Mach and non-Mach dynamics

Both $\mathcal{A}_M[\Omega; g, u]$ and $\mathcal{A}_G[\Omega; g]$ remain unchanged under all diffeomorphisms of spacetime $M$ onto itself. For the infinitesimal diffeomorphism $\phi : M \longrightarrow M$, let

$$\widetilde{g} =: \phi_* g, \ \widetilde{u} =: \phi_* u, \ \widetilde{x} = x \circ \phi$$

and

$$\overline{\delta}g = \widetilde{g} - g, \ \overline{\delta}u = \widetilde{u} - u, \ \delta x^\mu = \widetilde{x}^\mu(p) - x^\mu(p) = x^\mu(\phi(p)) - x^\mu(p)$$

then the change of $\mathcal{A}_M[\Omega; g, u]$ (or $\mathcal{A}_G[\Omega; g]$) can be divided into 2 parts: the part due to small change of the integrand and the part due to the small change of the integration domain

$$\mathcal{A}_G[\phi(\Omega); \widetilde{g}] - \mathcal{A}_G[\Omega; g] = \int_{\phi(\Omega)} d^4x \sqrt{-|\widetilde{g}(x)|} \frac{c^3}{16\pi G} R(\widetilde{g}(x), \partial\widetilde{g}(x), \partial^2\widetilde{g}(x))$$

$$- \int_\Omega d^4x \sqrt{-|g(x)|} \frac{c^3}{16\pi G} R(g(x), \partial g(x), \partial^2 g(x))$$

$$= \int_\Omega d^4x \sqrt{-|g(x)|} \frac{c^3}{16\pi G} (\frac{1}{2} R g^{\alpha\beta}(x) - R^{\alpha\beta}) \overline{\delta} g_{\alpha\beta}(x)$$

$$+ \int_\Omega d^4x \frac{\partial}{\partial x^\lambda} [\sqrt{-|g(x)|} \frac{c^3}{16\pi G} W^\lambda(x)]$$

$$+ \int_{\partial\Omega} ds_\lambda(x) \delta x^\lambda \sqrt{-|g(x)|} \frac{c^3}{16\pi G} R = 0 \tag{43}$$

By using Gauss theorem, we obtain

$$\int_{x(\Omega)} d^4x \sqrt{-|g(x)|} \frac{c^3}{16\pi G} (\frac{1}{2} R g^{\alpha\beta}(x) - R^{\alpha\beta}) \overline{\delta} g_{\alpha\beta}(x)$$

$$+ \int_{x(\Omega)} d^4x \frac{\partial}{\partial x^\lambda} [\sqrt{-|g(x)|} \frac{c^3}{16\pi G} (W^\lambda(x) + R\delta x^\lambda)] = 0 \tag{44}$$

Due to the arbitrariness of $\Omega$, one gets the following identities.

$$\sqrt{-|g(x)|} \frac{c^3}{16\pi G} [\frac{1}{2} R g^{\alpha\beta}(x) - R^{\alpha\beta}] \overline{\delta} g_{\alpha\beta}(x)$$

$$+ \frac{\partial}{\partial x^\lambda} [\sqrt{-|g(x)|} \frac{c^3}{16\pi G} (E^{\lambda\alpha\beta}(x) \overline{\delta} g_{\alpha\beta}(x) + F^{\lambda\gamma\alpha\beta}(x) \overline{\delta} \partial_\gamma g_{\alpha\beta}(x) + R\delta x^\lambda)] = 0 \tag{45}$$

Similarly,

$$\mathcal{A}_M[\phi(\Omega); \widetilde{g}, \widetilde{u}] - \mathcal{A}_M[\Omega; g, u] = \int_{\phi(\Omega)} d^4x \sqrt{-|\widetilde{g}(x)|} \mathcal{L}(\widetilde{g}(x), \widetilde{u}(x), \widetilde{\nabla}\widetilde{u}(x))$$

$$- \int_\Omega d^4x \sqrt{-|g(x)|} \mathcal{L}(g(x), u(x), \nabla u(x))$$

$$= \int_\Omega d^4x \sqrt{-|g(x)|} \{ [\frac{\partial \mathcal{L}}{\partial u^\theta_\xi(x)} - \nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}] \overline{\delta} u^\theta_\xi(x)$$

$$+ [\frac{1}{2} g^{\alpha\beta}(x) \mathcal{L} + \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}(x)} - \nabla_\lambda U^{\lambda\alpha\beta}(x)] \overline{\delta} g_{\alpha\beta}(x) \}$$

$$+ \int_\Omega d^4x \frac{\partial}{\partial x^\lambda} \{ \sqrt{-|g(x)|} [\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)} \overline{\delta} u^\theta_\xi(x) + U^{\lambda\alpha\beta}(x) \overline{\delta} g_{\alpha\beta}(x) + \mathcal{L}\delta x^\lambda] \} = 0 \tag{46}$$

Due to the arbitrariness of $\Omega$, one gets the following identities

$$\sqrt{-|g(x)|}\{[\frac{\partial \mathcal{L}}{\partial u_\xi^\theta(x)} - \nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)}]\overline{\delta} u_\xi^\theta(x) + \frac{1}{2}T^{\alpha\beta}(x)\overline{\delta} g_{\alpha\beta}(x)\}$$

$$+\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)}\overline{\delta} u_\xi^\theta(x) + U^{\lambda\alpha\beta}(x)\overline{\delta} g_{\alpha\beta}(x) + \mathcal{L}\delta x^\lambda]\} = 0 \qquad (47)$$

Eqns.(45) and (47) hold for all kinematically allowed movements $g$ and $u$ and all infinitesimal diffeomorphisms of $M$ onto itself $\phi$.

Combining eqns.(45) and (47), we get

$$\sqrt{-|g(x)|}\{\frac{c^3}{16\pi G}[\frac{8\pi G}{c^3}T^{\alpha\beta}(x) + \frac{1}{2}Rg^{\alpha\beta}(x) - R^{\alpha\beta}]\overline{\delta} g_{\alpha\beta}(x)$$

$$+[\frac{\partial \mathcal{L}}{\partial u_\xi^\theta(x)} - \nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)}]\overline{\delta} u_\xi^\theta(x)\} + \frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}$$

$$[\left(\frac{c^3}{16\pi G}E^{\lambda\alpha\beta}(x) + U^{\lambda\alpha\beta}(x)\right)\overline{\delta} g_{\alpha\beta}(x) + \frac{c^3}{16\pi G}F^{\lambda\gamma\alpha\beta}(x)\overline{\delta}\partial_\gamma g_{\alpha\beta}(x)$$

$$+\frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)}\overline{\delta} u_\xi^\theta(x) + \left(\frac{c^3}{16\pi G}R + \mathcal{L}\right)\delta_\sigma^\lambda \delta x^\sigma]\} = 0 \qquad (48)$$

Suppose $V$ is a smooth vector field on spacetime $M$, and denote by $G_V =: \{\phi_t : M \longrightarrow M \mid |t| < \epsilon\}$ the 1-parameter local group of diffeomorphisms of $M$ onto itself generated by $V$. Substituting $\overline{\delta}\ g = \phi_{t*}g - \phi_{0*}g = \phi_{t*}g\ -g$, $\overline{\delta}u = \phi_{t*}u - \phi_{0*}u = \phi_{t*}u - u$, $\delta x^\mu = x^\mu(\phi_t(p)) - x^\mu(\phi_0(p)) = x^\mu(\phi(p)) - x^\mu(p)$ into equation (48) and taking the derivatives with respect to $t$ at $t = 0$, we obtain

$$\sqrt{-|g(x)|}\{\frac{c^3}{16\pi G}[R^{\alpha\beta} - \frac{8\pi G}{c^3}T^{\alpha\beta}(x) - \frac{1}{2}Rg^{\alpha\beta}(x)](\mathcal{L}_V g)_{\alpha\beta}(x)$$

$$+[\nabla_\lambda \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)} - \frac{\partial \mathcal{L}}{\partial u_\xi^\theta(x)}](\mathcal{L}_V u)_\xi^\theta(x)\}$$

$$+\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[\left(\frac{c^3}{16\pi G}R + \mathcal{L}\right)\delta_\sigma^\lambda V^\sigma(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u_\xi^\theta(x)}(\mathcal{L}_V u)_\xi^\theta(x)$$

$$-\left(\frac{c^3}{16\pi G}E^{\lambda\alpha\beta}(x) + U^{\lambda\alpha\beta}(x)\right)(\mathcal{L}_V g)_{\alpha\beta}(x)$$

$$-\frac{c^3}{16\pi G}F^{\lambda\gamma\alpha\beta}(x)\partial_\gamma(\mathcal{L}_V g)_{\alpha\beta}(x)]\} = 0 \qquad (49)$$

where $\mathcal{L}_V$ denotes the Lie derivative with respect to vector field $V$. Eqn.(49) holds for all kinematically allowed $g$, $u$ and all smooth vector field $V$.

## 4.1 Noether's theorem for GR

For movements $(g, u)$ satisfying equations of motion of GR (38), (39), we get

$$\partial_\lambda \left[\sqrt{-|g(x)|} J^\lambda_{[V]}(x)\right] = 0, \text{ or } \nabla_\lambda J^\lambda_{[V]}(x) = 0$$

$$\text{or } \int_{\partial\Omega} ds_\lambda(x) \sqrt{-|g(x)|} J^\lambda_{[V]}(x) = 0 \tag{50}$$

where

$$J^\lambda_{[V]}(x) =: \left(\frac{c^3}{16\pi G} R + \mathcal{L}\right) \delta^\lambda_\sigma V^\sigma(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)} (\mathcal{L}_V u)^\theta_\xi(x)$$

$$- \left(\frac{c^3}{16\pi G} E^{\lambda\alpha\beta}(x) + U^{\lambda\alpha\beta}(x)\right) (\mathcal{L}_V g)_{\alpha\beta}(x) - \frac{c^3}{16\pi G} F^{\lambda\gamma\alpha\beta}(x) \partial_\gamma (\mathcal{L}_V g)_{\alpha\beta}(x) \tag{51}$$

is a vector field. Therefore eqn.(50) is the continuity equation for some conserved scalar. We see, all the Noether's conserved quantities in GR (not all conserved quantities) in GR are scalars. This is reasonable and natural geometrically, because we can not add up $(r, s)$-tensors distributed over different spacetime points, hence we can not talk about the change of the sum $(r, s)$-tensors over a spacelike hyper-surface unless $r = s = 0$. Here we saw once more that how harmoniously geometry and physics work together.

The expressions in the first two square brackets of eqn.(49) are independent of $V$, therefore the entirety of all Noether's continuity equations imply the equations of motion of GR conversely. It contains all the dynamic information of the system. In particular, it entails all conservation laws in GR no matter how they are derived (say, by using Noether's theorem or not).

## 4.2 Noether's theorem for non-Mach dynamics

For the pre-given spacetime metric $g$ and the matter movement $u$ satisfying equations of motion of non-Mach dynamics (41), eqn.(49) gives

$$\sqrt{-|g(x)|}\{\frac{c^3}{16\pi G}[R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}(x) - \frac{8\pi G}{c^3} T^{\alpha\beta}(x)](\mathcal{L}_V g)_{\alpha\beta}(x)$$

$$+ \frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[\left(\frac{c^3}{16\pi G} R + \mathcal{L}\right) \delta^\lambda_\sigma V^\sigma(x) - \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)} (\mathcal{L}_V u)^\theta_\xi(x)$$

$$- \left(\frac{c^3}{16\pi G} E^{\lambda\alpha\beta}(x) + U^{\lambda\alpha\beta}(x)\right) (\mathcal{L}_V g)_{\alpha\beta}(x) - \frac{c^3}{16\pi G} F^{\lambda\gamma\alpha\beta}(x) \partial_\gamma (\mathcal{L}_V g)_{\alpha\beta}(x)]\} = 0 \tag{52}$$

This is not a continuity equation for all infinitesimal diffeomorphisms of spacetime $M$ onto itself. For a specific non-Mach dynamics, that is, for a specific pre-given metric field $g$, denote by $\mathcal{K}_g$ the set of all Killing vector fields of $g$. If $\underline{V} \in \mathcal{K}_g$,that is if $\mathcal{L}_{\underline{V}}g = 0$, then from eqn.(52) we have

$$\frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}J^\lambda_{[\underline{V}]}(x)] = 0, \text{or } \nabla_\lambda J^\lambda_{[\underline{V}]}(x) = 0$$

$$\text{or } \int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}J^\lambda_{[\underline{V}]}(x) = 0$$

where

$$J^\lambda_{[\underline{V}]}(x) = \left(\frac{c^3}{16\pi G}R + \mathcal{L}\right)\delta^\lambda_\sigma \underline{V}^\sigma(x) - \frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}(\mathcal{L}_{\underline{V}}u)^\theta_\xi(x)$$
$$-\left(\frac{c^3}{16\pi G}E^{\lambda\alpha\beta}(x) + U^{\lambda\alpha\beta}(x)\right)(\mathcal{L}_{\underline{V}}g)_{\alpha\beta}(x) - \frac{c^3}{16\pi G}F^{\lambda\gamma\alpha\beta}(x)\partial_\gamma(\mathcal{L}_{\underline{V}}g)_{\alpha\beta}(x) \tag{53}$$

which can be further written as

$$J^\lambda_{[\underline{V}]}(x) = \left(\frac{c^3}{16\pi G}R + \mathcal{L}\right)\delta^\lambda_\sigma \underline{V}^\sigma(x) - \frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}(\mathcal{L}_{\underline{V}}u)^\theta_\xi(x)$$

For the specific non-Mach dynamics SR, we have

$$\frac{\partial}{\partial x^\lambda}\{\sqrt{-|g(x)|}[\mathcal{L}\delta^\lambda_\sigma \underline{V}^\sigma(x) - \frac{\partial\mathcal{L}}{\partial\nabla_\lambda u^\theta_\xi(x)}(\mathcal{L}_{\underline{V}}u)^\theta_\xi(x)]\} = 0$$

When $\{y^0, y^1, y^2, y^3\}$ is an inertial coordinate system, we get

$$\frac{\partial}{\partial y^\lambda}\{\ \mathcal{L}(\eta, u(y), \partial u(y))\delta^\lambda_\sigma \underline{V}^\sigma(y) - \frac{\partial\mathcal{L}(\eta, u(y), \partial u(y))}{\partial\partial_\lambda u^\theta_\xi(y)}(\mathcal{L}_{\underline{V}}u)^\theta_\xi(y)\ \} = 0 \tag{54}$$

Comparing Noether's conservation currents in GR and in non-Mach dynamics (51) and (53), one sees that they are alike. The difference seems to be that the $V$ in (51) can be any vector field on spacetime while the $\underline{V}$ in (53) can only be a Killing vector field for the pre-given metric $g$. But do not jump to the conclusion that for a specific non-Mach dynamics (for a specific pre-given metric $g$), the Noether's conserved currents are part of those in GR. Because the spacetime metric in a specific non-Mach dynamics is pre-given, while the Lorentzian metric field of spacetime in GR is to be determined by equations of motion (by the least action principle). One can not identify a vector field on the former with any vector field on the latter.

All the results obtained so far are the direct consequence of the least action principle and action (26). We will use them along with some sound facts from geometry to explore the pseudotensor-nonlocalizability problem, energy-momentum conservation in GR and gravitational energy-momentum in the following.

# 5 Pseudotensor, non-localizability problem in GR

## 5.1 How did the pseudotensor problem occur

In the above section we showed all the Noether's conserved quantities (not all conserved quantities) are scalars by using the active view-point in symmetry analyzing. In order to show how the pseudotensor problem occurs in GR, let us switch to the passive view-point.

Suppose $\{x\}$ is a given coordinate system and $\rho$ is a given index. The vector field

$$V : V(p) = (\frac{\partial}{\partial x^\rho})_p, \forall p \in M, \tag{55}$$

is expressed in coordinate system $\{x\}$ as $V^\alpha(x) = \delta^\alpha_\rho, \forall \alpha = 0, 1, 2, 3$. The 1-parameter local group of diffeomorphisms of $M$ onto itself generated by $V$, $G_{\frac{\partial}{\partial x^\rho}} = \{\phi_t : M \to M|\ t \in (-\epsilon, \epsilon)\}$, is described in coordinate system $\{x\}$ as $x^\alpha(\phi_t(p)) = x^\alpha(p) + t\delta^\alpha_\rho$, and we have

$$\delta x^\alpha = t\delta^\alpha_\rho,\ \overline{\delta} u^\theta_\xi(x) = -t\partial_\rho u^\theta_\xi(x),\ \overline{\delta} g_{\alpha\beta}(x) = -t\partial_\rho g_{\alpha\beta}(x). \tag{56}$$

The conservation law due to coordinate shift invariance is

$$\partial_\lambda[\sqrt{-|g(x)|}\tau^\lambda_\rho(x)] = 0, \tag{57}$$

where

$$\tau^\lambda_\rho(x) = \left(\frac{c^3}{16\pi G}R + \mathcal{L}\right)\delta^\lambda_\rho - \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(x)}\partial_\rho u^\theta_\xi(x)$$

$$-\left(\frac{c^3}{16\pi G}E^{\lambda\alpha\beta}(x) + U^{\lambda\alpha\beta}(x)\right)\partial_\rho g_{\alpha\beta}(x) - \frac{c^3}{16\pi G}F^{\lambda\gamma\alpha\beta}(x)\partial_\rho\partial_\gamma g_{\alpha\beta}(x) \tag{58}$$

is called the canonical energy-momentum. It is easy to check that

$$\tau^\mu_\sigma(y) \neq \frac{\partial y^\mu}{\partial x^\lambda}\frac{\partial x^\rho}{\partial y^\sigma}\tau^\lambda_\rho(x) \tag{59}$$

and this is referred to as the pseudotensor problem.

In an arbitrary coordinate system $\{y\}$, the vector field (55) $V = \frac{\partial}{\partial x^\rho}$ is expressed as

$$V^\mu(y) = \frac{\partial y^\mu}{\partial x^\rho}, \forall \mu = 0, 1, 2, 3. \tag{60}$$

and the 1-parameter local group generated by it, $G_{\frac{\partial}{\partial x^\rho}}$, is described as

$$y^\mu(\phi_t(p)) = y^\mu(p) + t\frac{\partial y^\mu}{\partial x^\rho}, \forall \mu = 0, 1, 2, 3. \tag{61}$$

and we have

$$(\mathcal{L}_{\frac{\partial}{\partial x^\rho}} u)^\theta_\xi(y) = \frac{\partial y^\mu}{\partial x^\rho}\frac{\partial}{\partial y^\mu}u^\theta_\xi(y) + \frac{\partial}{\partial y^\varphi}\left(\frac{\partial y^\theta}{\partial x^\rho}\right)u^\varphi_\xi(y) - \frac{\partial}{\partial y^\xi}\left(\frac{\partial y^\eta}{\partial x^\rho}\right)u^\theta_\eta(y)$$

$$(\mathcal{L}_{\frac{\partial}{\partial x^\rho}} g)_{\alpha\beta}(y) = V^\mu(y)\partial_\mu g_{\alpha\beta}(y) + \frac{\partial}{\partial y^\alpha}\left(\frac{\partial y^\mu}{\partial x^\rho}\right) g_{\mu\beta}(y) + \frac{\partial}{\partial y^\beta}\left(\frac{\partial y^\mu}{\partial x^\rho}\right) g_{\alpha\mu}(y) \tag{62}$$

And the corresponding conservation law is

$$\frac{\partial}{\partial y^\lambda}[\sqrt{-|g(y)|}J^\lambda_{[\frac{\partial}{\partial x^\rho}]}(y)] = 0, \tag{63}$$

where

$$J^\lambda_{[\frac{\partial}{\partial x^\rho}]}(y) = \left(\frac{c^3}{16\pi G}R + \mathcal{L}\right)\frac{\partial y^\lambda}{\partial x^\rho} - \frac{\partial \mathcal{L}}{\partial \nabla_\lambda u^\theta_\xi(y)}(\mathcal{L}_{\frac{\partial}{\partial x^\rho}} u)^\theta_\xi(y)$$
$$- \left(\frac{c^3}{16\pi G}E^{\lambda\alpha\beta}(y) + U^{\lambda\alpha\beta}(y)\right)(\mathcal{L}_{\frac{\partial}{\partial x^\rho}} g)_{\alpha\beta}(y) - \frac{c^3}{16\pi G}F^{\lambda\gamma\alpha\beta}(x)\partial_\gamma(\mathcal{L}_{\frac{\partial}{\partial x^\rho}} g)_{\alpha\beta}(y) \tag{64}$$

is a vector field defined by the specific coordinate system $\{x\}$ and index $\rho$. It is worth noting that this expression is good for all coordinate systems $\{y\}$ of $M$, including the specific coordinate system $\{x\}$.

Now, when people say eqn.(59) shows $\tau^\lambda_\rho(x)$ is not a tensor, they are taking $\tau^\lambda_\rho(x)$ and $\tau^\mu_\sigma(y)$ for components in coordinate systems $\{x\}$ and $\{y\}$ of the same geometrical, physical object $\tau$, and comparing them. But we think, Noether's conservation currents $J_{[\frac{\partial}{\partial x^\rho}]}$ and $J_{[\frac{\partial}{\partial y^\sigma}]}$ correspond to different 1-parameter local groups $G_{\frac{\partial}{\partial x^\rho}}$ and $G_{\frac{\partial}{\partial y^\sigma}}$ respectively, hence should not be taken for the same geometrical, physical object, even though the expression in $\{x\}$ for $J_{[\frac{\partial}{\partial x^\rho}]}$ and the expression in $\{y\}$ for $J_{[\frac{\partial}{\partial y^\sigma}]}$

$$J^\lambda_{[\frac{\partial}{\partial x^\rho}]}(x) = \tau^\lambda_\rho(x), \forall\lambda = 0, 1, 2, 3, \tag{65}$$

and

$$J^\mu_{[\frac{\partial}{\partial y^\sigma}]}(y) = \tau^\mu_\sigma(y), \forall\mu = 0, 1, 2, 3. \tag{66}$$

look the same. Conversely, $J^\lambda_{[\frac{\partial}{\partial x^\rho}]}(x)(= \tau^\lambda_\rho(x))$ and $J^\lambda_{[\frac{\partial}{\partial x^\rho}]}(y)(\neq \tau^\lambda_\rho(y))$ (when $\{x\}$, $\{y\}$ are different) are not alike, but they are referred to the same geometrical, physical object, and we have

$$J^\lambda_{[\frac{\partial}{\partial x^\rho}]}(y) = \frac{\partial y^\lambda}{\partial x^\kappa} J^\kappa_{[\frac{\partial}{\partial x^\rho}]}(x)$$

## 5.2 The principle of general relativity revisited

The first pseudotensor in GR was Einstein's gravitational energy-momentum $t^{\alpha\beta}(x)$ in eqn.(6). He derived equation (6) from his field equation before Noether's theorem was proposed. His derivation was correct, but not his physical interpretation. Because $t^{\alpha\beta}(x)$ and $t^{\alpha\beta}(y)$ had the same form, they were taken for components in coordinate systems $\{x\}$ and $\{y\}$ of the same geometrical, physical object $t$, according to the principle of general relativity. And this causes the pseudotensor-nonlocalizability problem. We are going to show this was a misunderstanding of the principle of general relativity. Einstein's gravitational energy-momentum $t^{\alpha\beta}(x)$ and $t^{\alpha\beta}(y)$ are not the same physical object, when coordinate systems $\{x\}$, $\{y\}$ are different.

According to Einstein, "What we call physics comprises that group of natural sciences which base their concepts on measurements; and whose concepts and propositions lend themselves to mathematical formulation. Its realm is accordingly defined as that part of the sum total of our knowledge which is capable of being expressed in mathematical terms." Therefore, to study physical processes, one has to choose some reference coordinate system first. The physical laws are objective. If their expressions depend on the reference coordinate systems chosen by individuals, they are certainly not being formulated properly. Therefore the principle of general relativity requires all the physical laws be expressed in different reference coordinate systems the same way. It is important, however, to distinguish general physical laws and concrete physical processes (or concrete physical quantities). The principle of general relativity also requires any concrete physical process be observed (or any concrete physical quantity be measured) from different reference coordinate systems the same way (All the reference coordinate systems are the same good for observing and measuring). However, this does not mean that a concrete physical process (or a concrete physical quantity) should have the same relation to different reference coordinate systems. In general, any proposition in physics, no matter a common physical law or a concrete physical process, true or false, can be formulated in all coordinate systems the same way. For the only Mach dynamics GR (Its spacetime metric is to be determined by the least action principle), if a proposition properly formulated in terms of coordinates $\{x\}$ as $\mathcal{P}[x]$ is true, then for any coordinates $\{y\}$, $\mathcal{P}[y]$ is true.

To illustrate the above idea, let us consider the following examples.

Equations of motion are common physical laws. They are determined by the least action principle. So the action of a dynamic system should be expressed the same way in all coordinate systems, and should be an invariant under general coordinate transformations.

Einstein's field equation (2) is a general law of physics. It has the same form in all reference coordinate systems. The matter energy-momentun tensor $T^{\alpha\beta}(x)$ is part of Einstein's field equation, hence it has the same form in all reference coordinate systems. For a given dynamic system in GR, there is only one energy-momentun tensor of matter $T$, which is a symmetrical (2,0)-tensor field, independent of coordinates. For any fixed coordinate system $\{x\}$ and index $\rho$, the conservation law due to coordinate shift ($\delta x^\alpha = t\delta^\alpha_\delta$) invariance can be observed from all coordinate system $\{y\}$ in the same way (64), even though its relation with coordinate system $\{x\}$ (58) (note that $\tau^\lambda_\rho(x) = J^\lambda_{\left[\frac{\partial}{\partial x^\rho}\right]}(x)$) is special. Such conserved quantities are called canonical energy-momentun (they are actually scalars). There are infinitely many canonical energy-momentums for one dynamic system, because there are infinitely many different generating vector field $\frac{\partial}{\partial x^\rho}{}'$s.

One more illustration is the example (3) on p.252 in Ohanian's book[19]. We rewrite it as, in a specified coordinate system $\{\xi^0, \xi^1, \xi^2, \xi^3\}$, tangent field $A$ and cotangent field $B$ satisfy the following equation

$$A^\mu(\xi)|_p = B_\mu(\xi)|_p, \forall \mu = 0, 1, 2, 3; p \in M \tag{67}$$

This proposition is not a common physical law. But we can still describe it in terms of any coordinate system $\{x^0, x^1, x^2, x^3\}$ (including coordinate system $\{\xi^0, \xi^1, \xi^2, \xi^3\}$) in a unified (coordinate free) way

$$\frac{\partial \xi^\mu}{\partial x^\lambda} A^\lambda(x)|_p = B_\lambda(x) \frac{\partial x^\lambda}{\partial \xi^\mu}|_p, \forall \mu = 0, 1, 2, 3; p \in M \tag{68}$$

even though its relation to coordinate system $\{\xi^0, \xi^1, \xi^2, \xi^3\}$ (67) is special.

## 5.3 More observations from geometry

The following geometrical facts would show all continuity equations in GR are the conservation laws for scalars, no matter derived by using Noether's theorem or not.

**Proposition 2** *Suppose $t$ is an $(r+1, s)$-tensor field on spacetime $M$, ($\xi^0$, $\xi^1$, $\xi^2$, $\xi^3$) is a given coordinate system of $M$. and $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ are some given indices. If for all spacetime region $\Omega$*

$$\int_{\partial\Omega} ds_\lambda(\xi)\sqrt{-|g(\xi)|}\, t^{\lambda\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}(\xi) = 0$$

*or equivalently*

$$\frac{\partial}{\partial \xi^\lambda}[\sqrt{-|g(\xi)|}\, t^{\lambda\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}(\xi)]|_p = 0, \forall p \in M \tag{67}$$

*then for every coordinate system $(x^0, x^1, x^2, x^3)$ of $M$,*

$$\frac{\partial}{\partial x^\lambda}[\sqrt{-|g(x)|}\, j^\lambda(x)] = 0 \tag{68}$$

$$\textit{or equivalently} \int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}\, j^\lambda\,(x) = 0$$

*where $j$ is a tangent field on $M$ defined as follows*

$$j(dx^\lambda) = t(dx^\lambda, d\xi^{\alpha_1}, \ldots, d\xi^{\alpha_r}, \frac{\partial}{\partial \xi^{\beta_1}}, \ldots, \frac{\partial}{\partial \xi^{\beta_s}}), \forall \{x\} \textit{ and } \lambda \tag{69}$$

**Proof.**

$$\frac{\partial}{\partial \xi^\lambda}\, [\sqrt{-|g(\xi)|}\, t^{\lambda\alpha_1\cdots\alpha_r}{}_{\beta_1\cdots\beta_s}(\xi)] = \frac{\partial}{\partial \xi^\lambda}\, [\sqrt{-|g(\xi)|}\, j^\lambda(\xi)]$$

$$= \frac{\partial}{\partial\, \xi^\lambda}[\left|\frac{\partial x}{\partial \xi}\right| \frac{\partial \xi^\lambda}{\partial x^\mu}\sqrt{-|g(x)|}\, j^\mu(x)\ ]$$

$$= \left|\frac{\partial x}{\partial \xi}\right| \frac{\partial \xi^\lambda}{\partial x^\mu} \frac{\partial}{\partial\, \xi^\lambda}[\sqrt{-|g(x)|} j^\mu(x)\ ] + \frac{\partial}{\partial \xi^\lambda}[\left|\frac{\partial x}{\partial \xi}\right| \frac{\partial \xi^\lambda}{\partial x^\mu}] \sqrt{-|g(x)|} j^\mu(x)$$

$$= \left|\frac{\partial x}{\partial \xi}\right| \frac{\partial}{\partial x^\mu}[\sqrt{-|g(x)|} j^\mu(\ x\ )]. \tag{70}$$

$$\because \frac{\partial}{\partial \xi^\lambda}[\left|\frac{\partial x}{\partial \xi}\right| \frac{\partial \xi^\lambda}{\partial x^\mu}] = 0, \forall \mu = 0, 1, 2, 3 \tag{71}$$

■

**Corollary 3** *When $r = s = 0$, preposition 1 tells us: Suppose $j$ is a vector field on spacetime $M$, and $(\xi^0, \xi^1, \xi^2, \xi^3)$ is a given coordinate system of $M$. if*

$$\frac{\partial}{\partial \xi^\lambda}\ [\sqrt{-|g(\xi)|} j^\lambda(\xi)] = 0$$

*then for all coordinate systems $(x^0, x^1, x^2, x^3)$*

$$\frac{\partial}{\partial x^\lambda}\ [\sqrt{-|g(x)|} j^\lambda(x)] = 0 \tag{72}$$

Let us get back to eqn.(6). For a specified pair of $\{x\}$ and $\mu$, define vector field $J$

$$J^\lambda(y) = T(dy^\lambda, dx^\mu) + \frac{\partial y^\lambda}{\partial x^\sigma} t^{\sigma\mu}(x), \forall\ \{y\} \text{ and } \mu \tag{73}$$

Then we have the following conservation law for some scalar defined by coordinate system $\{x\}$ and index $\mu$.

$$\frac{\partial}{\partial y^\lambda}[\sqrt{-|g(y)|} J^\lambda(y)] = 0, \forall \text{ coordinate systems } \{y\} \text{ of } M \tag{74}$$

Therefore, eqn.(6) plus each pair of $\{x\}$ and $\mu$, determines a conservation law of a scalar. We have infinitely many such conserved scalars. Comparing eqn.(73) and $T^{\alpha\beta}(y) + t^{\alpha\beta}(y)$, one sees the former is addition of two vector fields, while the latter is considered addition of a tensor and a pseudotensor field. So, the new perspective enables us to get rid of the embarrassing situation: accepting the addition of a tensor and a pseudotensor, which is absurd geometrically.

### 5.4 Is non-localizability of gravitational energy a consequence of equivalence principle?

In their famous book [6], Misner et al. argued, "One can always find in any given locality a frame of reference in which all local 'gravitational fields' (all Christoffel symbols; all $\Gamma^\alpha_{\mu\nu}$) disappear. No Γ's means no 'gravitational field'. No local gravitational field means no 'local gravitational energy-momentum'." Then they claimed, "It (gravitational energy) is not localizable. The equivalence principle forbids."

Inertial mass and gravitational mass are identified experimentally as the same physical quantity. If all reference coordinate systems are the same good for describing physical processes, then inertial force and gravitational force are indistinguishable, and should be taken as the same thing. This enlightened Einstein to realize gravity is a manifestation of spacetime bending and finally to establish his general theory of relativity. A metric field contains all the geometrical information of a generalized Riemannian manifold. It can completely describe spacetime bending in GR. Therefore in GR, we have only spacetime metric and variables describing matter movement; we do not have gravitational force, inertial force and equivalence principle such pre-GR concepts any more. The spacetime metric has taken all their places in GR. They were the mid-wife of the infant GR, they are not part of GR as Synge said [20].

"A free particle's world line is a timelike geodesic in GR" can be looked upon as "it moves under gravity according to Newton's law of motion in a flat spacetime." This is just approximately correct at the weak field-low speed limit. The geometrical description of gravitation is not equivalent to the force field description (or action at a distance description). The latter is just an alternative way of looking upon the matter, approximately effective at the weak field-low speed limit, in some aspect; but it is not correct after all.

The concept of energy evolves with people's concept of physical reality. Newton's physical reality is, the world is composed of mass points interacting with instantaneous action at a distance. The mass points move according to Newton's laws of motion in absolute space and absolute time. Accordingly, the position related potential energy $V(...\overrightarrow{r_i}...)$ of mass point system interacting with a conservative action at a distance is determind by

$$\overrightarrow{f_i} = -\frac{\partial}{\partial \overrightarrow{r_i}} V(...\overrightarrow{r_i}...), \forall i = 1, 2 \dots \tag{76}$$

to an integration constant. One can talk about negative potential energy. In the 19th century, Faraday, Maxwell and Hertz introduced the concept of force field into physics. Action at a distance, absolute space and absolute time withdrew from the historical stage of physics forever. Correspondingly electrical field energy took the place of Coulomb potential energy. The energy density of static electrical field is

$$\varepsilon_E(\overrightarrow{r}) = \frac{1}{8\pi} \left| \overrightarrow{E}(\overrightarrow{r}) \right|^2$$

If treating Newton's universal gravity the same way as treating Colomb force, the energy density of gravitational field would be

$$\varepsilon_G(\overrightarrow{r}) = \frac{-1}{8\pi} \left| \overrightarrow{g}(\overrightarrow{r}) \right|^2$$

This is not allowed by Einstein's mass-energy relation and thermodynamics. It shows we can not take gravity as a force field.

The "gravitational field" in [6] can disappear merely due to switching coordinates, while the spacetime metric and variables describing matter movements remain unchanged. Mere coordinate transformations can not change any real things (physical or geometrical), say, a particle's world line, geodesics, connections, curvatures, etc. Consider a sphere in $\mathbb{R}^3$ (the surface of the earth). When we use the latitude and longitude as local coordinates, all the $\Gamma$'s disappear on the equator. This does not make points on the equator any different from other points on the sphere. So, the "gravitational field" as a force field in [6] is not an objective physical, geometrical concept.

Now let us get to the question, how to understand conservation of energy-momentun in GR?

# 6 How to understand conservation of energy-momentum in GR

As shown in section 2, in a curved spacetime, $(r, s)$-tensors distributed at different spacetime points can not be added up unless $r = s = 0$. In particular, the sum 4-vector of matter energy-momentum over a hyper-surface does not make sense. Then we can not understand conservation law of matter energy-momentum in GR the same way as in SR. Fortunately, in a curved spacetime, we can still talk about the density-flux tensor of matter energy-momentum at a spacetime point. That is, we can talk about the sum 4-vector of energy-momentum distributed over an infinitesimal hypersurface, we can alk about the amount of matter energy-momentum created in an infinitesimal 4-volume, when neglecting higher order infinitesimal deviations.

Now, for any point $p$ in spacetime, choose a local inertial coordinate system $\{x\}$ ($g_{\alpha\beta}(x)|_p = \eta_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}(x)|_p = 0$, hence the deviation $g_{\alpha\beta}(x)|_q - \eta_{\alpha\beta}$ is the second order infinitesimal for nearby point $q$) and integrate $\sqrt{-|g(x)|}\nabla_\lambda T^{\lambda\mu}(x) = 0$ over an infinitesimal neighborhood of $p$, bounded by an infinitesimal past spacelike hyper-surface $\triangle\Sigma$, an infinitesimal future spacelike hyper-surface $\triangle\Sigma'$ and an infinitesimal timelike hyper-surface $\Delta\Gamma$ which links the boundaries of $\triangle\Sigma$ and $\triangle\Sigma'$. Using the mid-value theorem and neglecting higher order infinitesimal error, we obtain in this nearly flat coordinate system $\{x\}$

$$\begin{aligned} & ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)|_{\triangle\Sigma'} - ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)|_{\triangle\Sigma} \\ = \quad & -ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)|_{\triangle\Gamma}, \forall\mu = 0, 1, 2, 3 \qquad (75) \end{aligned}$$

This tells us, the continuity equation of matter energy-momentum still holds in GR, but it holds only for all infinitesimal spacetime regions. $\nabla_\lambda T^{\lambda\mu}(x) = 0$ means there is no spring and sink of matter energy-momentum everyhere in spacetime. It does not ruin the law of conservation of matter energy-momentum in GR, but it itself is the law of conservation of matter energy-momentum in GR. It was unnecessary and improper to introduce "the gravitational energy-momentum $t^{\lambda\mu}(x)$" to save the law of conservation of energy-momentum in GR, which has caused confusions like pseudotensors, non-localizability that has lasted for nearly 100 years.

We are now in a position to explore the question: whether gravitational field carries energy-momentum or not.

## 7 Gravitational field does not carry energy-momentum

When we say the electromagnetic field carries energy-momentum, we mean it exchanges energy-momentum with other matter. In other words, if some thing does not exchange energy-momentum with all other things under any circumstances, we say it does not carry energy-momentum. "Interacting with force" means "exchanging energy-momentum". Therefore, "the electromagnetic field carries energy-momentum" means it is a force field. In the following we will study the case of classical electrodynamics in GR, and show gravitational field does not exchange energy-momentum with particles (charged and uncharged) and electromagnetic field.

Suppose the dynamic system consists of particles, electromagnetic field $A$ and spacetime metric field $g$. Denote the world line of a particle with charge $q$ and rest mass $m$ by $\gamma : \mathbb{R} \longrightarrow M$. In coordinate system $\{x\}$, the parameter equation of $\gamma$ is $x^\alpha = x^\alpha(\gamma(\tau)) =: X^\alpha(\tau), \forall \tau \in \mathbb{R}$, where the parameter $\tau$ is the proper time. We use the following action integral over spacetime region $\Omega$

$$\mathcal{A}[\Omega; g, A, \gamma] = \mathcal{A}_m[\Omega; g, \gamma] + \mathcal{A}_q[\Omega;\ A, \gamma] + \mathcal{A}_{EM}[\Omega; g, A\ ] + \mathcal{A}_G[\Omega; g\ ] \quad (76)$$

where

$$\mathcal{A}_m[\Omega; g, \gamma] = \sum \int_{\gamma(\tau)\in\Omega} d\tau\,(-mc)\sqrt{-g_{\alpha\beta}(X(\tau))\frac{dX^\alpha(\tau)}{d\tau}\frac{dX^\beta(\tau)}{d\tau}}, \quad (77)$$

is the action for the particles, and $\sum$ means summing up over all particles;

$$\mathcal{A}_q[\Omega;\ A, \gamma] = \int_\Omega d^4x\sqrt{-|g(x)|}\frac{1}{c^2}j^\alpha(x)A_\alpha(x)$$

$$= \int_\Omega d^4x\sqrt{-|g(x)|}\frac{1}{c^2}\sum\int_{\mathbb{R}} d\tau cq\frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}}\frac{dX^\alpha(\tau)}{d\tau}A_\alpha(x)$$

$$= \sum\int_{\gamma(\tau)\in\Omega} d\tau\frac{q}{c}\frac{dX^\alpha(\tau)}{d\tau}A_\alpha(X(\tau)), \quad (78)$$

is the action for interaction between charged particles and electromagnetic field;

$$\mathcal{A}_{EM}[\Omega; g, A\ ] = \int_\Omega d^4x\sqrt{-|g(x)|}$$

$$\frac{-1}{16\pi c}g^{\alpha\mu}(x)g^{\beta\nu}(x)\left(\nabla_\alpha A_\beta(x) - \nabla_\beta A_\alpha(x)\right)\left(\nabla_\mu A_\nu(x) - \nabla_\nu A_\mu(x)\right), \quad (79)$$

is the action for electromagnetic field, and

$$\mathcal{A}_G[\Omega; g\ ] = \int_\Omega d^4x\sqrt{-|g(x)|}\frac{c^3}{16\pi G}R\left(g(x), \partial g(x), \partial^2 g(x)\right). \quad (80)$$

is the action for gravitational field.

Consider the difference of actions of two kinematically allowed movements close to each other.

$$\overline{\delta}\mathcal{A}_m\left[\Omega;g,\gamma\right]=\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{m}{2}\frac{dX^{\alpha}(\tau)}{d\tau}\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}g_{\alpha\beta}\left(X(\tau)\right)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{1}{2}\partial_{\gamma}g_{\alpha\beta}\left(X(\tau)\right)m\frac{dX^{\alpha}(\tau)}{d\tau}\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{d}{d\tau}\left[g_{\alpha\beta}\left(X(\tau)\right)m\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}X^{\alpha}(\tau)\right]$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\left(-1\right)g_{\gamma\beta}\left(X(\tau)\right)\frac{d}{d\tau}\left(m\frac{dX^{\beta}(\tau)}{d\tau}\right)\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{-1}{2}\partial_{\alpha}g_{\gamma\beta}\left(X(\tau)\right)\frac{dX^{\alpha\cdot}(\tau)}{d\tau}\left[m\frac{dX^{\beta}(\tau)}{d\tau}\delta X^{\gamma}(\tau)\right]$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{-1}{2}\partial_{\beta}g_{\gamma\alpha}\left(X(\tau)\right)\frac{dX^{\alpha\cdot}(\tau)}{d\tau}\left[m\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}X^{\gamma}(\tau)\right]$$

$$=\int_{\Omega}d^4x\sqrt{-\left|g(x)\right|}\frac{1}{2}\sum\int_{\gamma}d\tau m\frac{dX^{\alpha}(\tau)}{d\tau}\frac{\delta^4\left(x-X(\tau)\right)}{\sqrt{-\left|g(x)\right|}}\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}g_{\alpha\beta}\left(x\right)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau-g_{\gamma\rho}\left(X(\tau)\right)\frac{d}{d\tau}\left(m\frac{dX^{\rho}(\tau)}{d\tau}\right)\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{-1}{2}g_{\gamma\rho}(X(\tau))\Gamma^{\rho}_{\alpha\beta}\left(X(\tau)\right)\left(m\frac{dX^{\beta}(\tau)}{d\tau}\right)\frac{dX^{\alpha\cdot}(\tau)}{d\tau}\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{d}{d\tau}\left[g_{\alpha\beta}\left(X(\tau)\right)m\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}X^{\alpha}(\tau)\right]$$

$$=\int_{\Omega}d^4x\sqrt{-\left|g(x)\right|}\frac{1}{2}T_m^{\alpha\beta}(x)\overline{\delta}g_{\alpha\beta}\left(x\right)+\sum\int_{\gamma(\tau)\in\Omega}d\tau\left(-1\right)g_{\gamma\rho}\left(X(\tau)\right)$$

$$\left[\frac{d}{d\tau}\left(m\frac{dX^{\rho}(\tau)}{d\tau}\right)+\Gamma^{\rho}_{\alpha\beta}\left(X(\tau)\right)\left(m\frac{dX^{\beta}(\tau)}{d\tau}\right)\frac{dX^{\alpha\cdot}(\tau)}{d\tau}\right]\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{d}{d\tau}\left[g_{\alpha\beta}\left(X(\tau)\right)m\frac{dX^{\beta}(\tau)}{d\tau}\overline{\delta}X^{\alpha}(\tau)\right]\tag{81}$$

where

$$T_m^{\alpha\beta}(x)=\sum\int_{\mathbb{R}}d\tau\frac{dX^{\alpha}(\tau)}{d\tau}\frac{\delta^4\left(x-X(\tau)\right)}{\sqrt{-\left|g(x)\right|}}m\frac{dX^{\beta}(\tau)}{d\tau}\tag{82}$$

is the particles' energy-momentun density-flux tensor field.

$$\overline{\delta}\mathcal{A}_q\left[\Omega;\ A,\gamma\right]=\sum\int_{\gamma(\tau)\in\Omega}d\tau\frac{q}{c}\frac{dX^{\alpha}(\tau)}{d\tau}\overline{\delta}A_{\alpha}\left(X(\tau)\right)$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \frac{q}{c}\frac{dX^{\alpha}(\tau)}{d\tau}\partial_{\gamma}A_{\alpha}\left(X(\tau)\right)\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \frac{d}{d\tau}\left[\frac{q}{c}A_{\alpha}\left(X(\tau)\right)\overline{\delta}X^{\alpha}(\tau)\right]$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \left(-1\right)\frac{q}{c}\frac{dX^{\alpha}(\tau)}{d\tau}\partial_{\alpha}A_{\gamma}\left(X(\tau)\right)\overline{\delta}X^{\gamma}(\tau)$$

$$= \int_{\Omega} d^4x\sqrt{-\left|g(x)\right|}\frac{1}{c}\sum \int_{\mathbb{R}} d\tau q \frac{\delta^4\left(x - X(\tau)\right)}{\sqrt{-\left|g(x)\right|}}\frac{dX^{\alpha}(\tau)}{d\tau}\overline{\delta}A_{\alpha}\left(x\right)$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \frac{q}{c}\frac{dX^{\alpha}(\tau)}{d\tau}\left[\nabla_{\gamma}A_{\alpha}\left(X(\tau)\right) - \nabla_{\alpha}A_{\gamma}\left(X(\tau)\right)\right]\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \frac{d}{d\tau}\left[\frac{q}{c}A_{\alpha}\left(X(\tau)\right)\overline{\delta}X^{\alpha}(\tau)\right]$$

$$= \int_{\Omega} d^4x\sqrt{-\left|g(x)\right|}\frac{1}{c^2}j^{\alpha}(x)\overline{\delta}A_{\alpha}\left(x\right)$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \frac{q}{c}\frac{dX^{\alpha}(\tau)}{d\tau}\left[\nabla_{\gamma}A_{\alpha}\left(X(\tau)\right) - \nabla_{\alpha}A_{\gamma}\left(X(\tau)\right)\right]\overline{\delta}X^{\gamma}(\tau)$$

$$+\sum \int_{\gamma(\tau)\in\Omega} d\tau \frac{d}{d\tau}\left[\frac{q}{c}A_{\alpha}\left(X(\tau)\right)\overline{\delta}X^{\alpha}(\tau)\right] \tag{82}$$

Because $\mathcal{A}_{EM}\left[\Omega; g, A\ \right]$ and $\mathcal{A}_{G}\left[\Omega; g\ \right]$ do not depend on $\gamma$'s, we obtain the equation of motion for particles' variables.

$$\frac{\delta\mathcal{A}\left[\Omega; g, A, \gamma\right]}{\delta X^{\gamma}(\tau)} = \frac{\delta\mathcal{A}_{m}\left[\Omega; g, \gamma\right]}{\delta X^{\gamma}(\tau)} + \frac{\delta\mathcal{A}_{q}\left[\Omega; A, \gamma\right]}{\delta X^{\gamma}(\tau)} = 0,$$

$$g^{\rho\gamma}\left(X(\tau)\right)\frac{q}{c}\left[\nabla_{\gamma}A_{\alpha}\left(X(\tau)\right) - \nabla_{\alpha}A_{\gamma}\left(X(\tau)\right)\right]\frac{dX^{\alpha}(\tau)}{d\tau}$$

$$= \frac{d}{d\tau}\left(m\frac{dX^{\rho}(\tau)}{d\tau}\right) + \Gamma^{\rho}_{\alpha\beta}\left(X(\tau)\right)\left(m\frac{dX^{\beta}(\tau)}{d\tau}\right)\frac{dX^{\alpha\cdot}(\tau)}{d\tau},$$

$$\frac{D}{d\tau}p^{\rho}(\tau) = g^{\rho\gamma}\left(X(\tau)\right)\frac{q}{c}F_{\gamma\alpha}(X(\tau))\frac{dX^{\alpha}(\tau)}{d\tau}, \forall\rho = 0, 1, 2, 3 \tag{83}$$

where $p^{\rho}(\tau) = m\frac{dX^{\rho}(\tau)}{d\tau}$ is the particle's energy-momentum 4-vector, defined on its world line $\gamma$, and the rhs of (83) is the covariant Lorentz 4-force exerted on the particle by the electromagnetic field (the energy-momentum that the electromagnetic field gives to the particle in per unit proper time). Keep in mind the meaning of " change of a particle's energy-momentum 4-vector" (See section 2). Eqn.(83) reads:

(i) if $q = 0$, the world line is a timelike geodesic, and the free particle's energy-momentum 4-vector does not change;

(ii) if $q \neq 0$, the change of a charged particle's energy-momentum 4-vector during $d\tau$ is just the amount that the electromagnetic field gives to it during $d\tau$.

Therefore, the gravitational field does not exchange energy-momentum with mass points no matter charged or not. It is totally different from electromagnetic field which exchanges energy-momentum with charges and currents constantly.

So far we know that in a small neighborhood, change of particles' energy-momentum are all from the electromagnetic field. but we don't know yet whether the electromagnetic field exchanges energy-momentum with the gravitational field or not. However, the conservation law of matter energy-momentum in GR

$$\nabla_\lambda T^{\lambda\mu}(x)|_p = \nabla_\lambda \left[T_m^{\lambda\mu}(x) + T_{EM}^{\lambda\mu}(x)\right]|_p = 0, \forall \mu = 0,1,2,3, p \in M \tag{84}$$

tells us, in any small neighborhood, all the energy-momentum the electromagnetic field gives away equals the amount the particles gain in this small neighborhood. Therefore, the electromagnetic field does not exchange energy-momentum with gravitational field.

The gravitational field does not exchange energy-momentum with both particles and electromagnetic field. So, it does not carry energy-momentum. it is not a force field. And gravity, the oldest natural force known to people, is not really a natural force.

All the experiments detecting energy carried by gravitational waves failed. LIGO's experiment detected the change of distance, it was a pure geometric measurement. It has nothing to do with gravitational energy radiation. It was just what I expected.

If the classical tensor analysis helped Einstein to establish his general theory of relativity, modern differential geometry are helping people to have deeper insights into GR. GR has been the most beautiful theory in physics, but it was messed by pseudotensors, non-localizability and gravitational energy-momentum which resides nowhere like a ghost. I believe modern geometry will finally help GR to recover its beauty.

# References


[1] A. Einstein, Sitzungsber preuss. Akad. Wiss. **2** (1916) 1111.

[2] H. Bauer, Physikalische Zeitschrift, **19** (1918) 163.

[3] I.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Butter-Worth-Heinemann, Beijing (1999).

[4] R.C. Tolman, Physical Rev. **35** (1930) 875 .

[5] C. Møller, The Theory of relativity, Oxford University Press, Oxford (1972).

[6] C. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation, Freeman, New York (1973).

[7] A. Komar, Physical Rev. **113** (1959) 934.

[8] R. Arnowitt, S. Deser, and C.W. Misner, The Dynamics of General Relativity, in Gravitation: A Introduction to Current Research, ed. Witten, L., Wiley, New York (1962).

[9] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, Proc. Roy. Soc. London A **269** (1962) 21.

[10] R. Schoen and S.-T. Yau, Commun. Math. Phys. **79** (1981) 231.

[11] E. Witten, Commun. Math. Phys. **80** (1981) 381.

[12] L.B. Szabados, Living Rev. Relativity **7** (2004) 4.

[13] R.P. Feynman, et al, Feynman Lectures on Gravitation, Westview Press, Boulder (2002).

[14] R.A. Hulse and J.H. Taylor Jr , Astrophys. J. (Lett.) **195** (1975) L51.

[15] B.P. Abbott, et al, PRL **116** (2016) 061102.

[16] H. Bondi, Proc. R. Soc. Lond. A **427** (1990) 249.

[17] J.M. Nester, et al, Dynamic geometry and Gravitational Energy, (2004).

[18] M.J. Gotay, et al, Momentum maps and classical fields, Part I: Covariant field theory. www.arxiv.org, physics/9801019v2, (2004).

[19] Hans C. Ohanian, Gravitation and Spacetime, W.W. Norton & company, New York (1976).

[20] J.L. Synge, Relativity: The General Theory, North-Holland Publishing Company, Amsterdam (1960).