Measurement of time by quantum clocks

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A clock is a dynamical system which passes through a succession of states at constant time intervals. If coupled to another system, it can measure the duration of a physical process and even keep a permanent record of it, such as in a time-of-flight experiment or in observing the lifetime of an unstable atom. A clock can also be used to control the duration of a process, e.g., the precession of a spin in a magnetic field which is turned on and off at prescribed times. This article shows how to construct time-independent Hamiltonians describing these possible uses of a quantum clock. As expected, a good time resolution entails a large energy exchange between the clock and the other system, thereby modifying the evolution of the latter. This evolution may even be halted by using a clock which is too precise (this is the quantum analog of Zeno's paradox).

I. INTRODUCTION

The measurement of time is different from that of quantities like position, energy, momentum, etc., because time is not a dynamical variable. For example, a classical particle may have a well-defined position, an energy, and so on, but we cannot say that "a particle has a well-defined time." Formally, the Poisson bracket (in classical mechanics) or commutator (in quantum theory) of t with any dynamical variable is always zero.

What we call the measurement of time actually is the observation of some dynamical variable, the law of motion of which is known (and is usually uniform, like the motion of a pointer on a clock dial). For example, if we observe a free particle with Hamiltonian $H = p^2/2m$, the quantity $T = mq/p$ has Poisson brackets $[T,H] = 1$ and can therefore be considered as a "realization of time" (for that Hamiltonian). In quantum theory, we could likewise define

$$T = \frac{i\hbar m}{2}\left(\frac{1}{p}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\frac{1}{p}\right),$$

which satisfies the commutation relation $[T,H] = i\hbar$, but T is not a convenient "time operator." In particular, the eigenfunctions of $T\psi = t\psi$, namely,

$$\psi \sim p^{1/2}\exp(-ip^2t/2m\hbar),$$

have no simple physical meaning.

If we take, instead of a free particle, a harmonic oscillator with $H = (1/2)(p^2 + q^2)$, the situation is even worse. In classical mechanics, we can define $T = \arctan(q/p)$ which is multiple valued, just like the hour on an ordinary watch. But there is no such thing as a "multiple valued operator" in a correctly defined Hilbert space.[1]

Section II of this paper discusses the construction of a quantum clock, according to the blueprint of Salecker and Wigner.[2] These authors were concerned mostly with the minimum mass required for a clock used in measuring spacetime distances. Here, our chief concern is different, namely, how much we are perturbing a system by coupling it to a physical clock. Intuitively, we expect that a very good time resolution can be obtained only at the expense of exchanging a large amount of energy with the measured system, thereby modifying the evolution of the latter.

Sections III–V give some examples of the use of quantum clocks. Section III describes a time-of-flight experiment and Sec. IV describes the measurement of the lifetime of an unstable atom. In Sec. V, a clock is used to control the duration of a process (rather than to measure it), in this case to turn on and off, at prescribed times, a magnetic field causing the precession of a spin. In all these examples, the Hamiltonian is time *independent* (the clock is part of a *closed* system). Our purpose is to show how this time-independent clock keeps a permanent record of the physical processes which it has monitored.

II. CONSTRUCTION OF A QUANTUM CLOCK

It is convenient (though not necessary) to assume that our clock has an odd number,

$$N = 2j + 1,$$

of states and to represent the latter as the wave functions

$$u_m(\theta) = (2\pi)^{-1/2}e^{im\theta}, \qquad m = -j, \ldots, j,$$

with $0 \le \theta < 2\pi$. Another orthogonal basis for the clock's wave functions can be

$$v_k(\theta) = N^{-1/2}\sum e^{-2\pi ikm/N}u_m$$
$$= (2\pi N)^{-1/2}\sin\left[\frac{N}{2}\left(\theta - \frac{2\pi k}{N}\right)\right]\Big/\sin\left[\frac{1}{2}\left(\theta - \frac{2\pi k}{N}\right)\right],$$

for $k = 0, \ldots, N - 1$. For large N, these functions have a sharp peak at $\theta = 2\pi k/N$ and can be visualized as "pointing to the kth hour" with an angle uncertainty $\pm\pi/N$.

We can then define projection operators

$$P_k v_m = \delta_{km} v_m,$$

and a "clock time" operator

$$T_c = \tau \sum k P_k,$$

where τ is the time resolution of our clock. The eigenvectors of T_c are the v_k and the corresponding eigenvalues are $t_k = k\tau$, with $k = 0, \ldots, N - 1$. Therefore measuring T_c can at best yield a discrete approximation to the true time (just like reading a digital watch). We shall always assume that the initial state of the clock is v_0.

The clock's Hamiltonian will be written as

$$H_c = \omega J,$$

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where

$$\omega = 2\pi/N\tau$$

and $J = -i\hbar\partial/\partial\theta$. Clearly

$$H_c u_m = m\hbar\omega u_m,$$

whence

$$\exp(-iH_c t/\hbar)u_m = e^{-im\omega t}u_m = (2\pi)^{-1/2}e^{im(\theta-\omega t)}.$$

It follows that

$$\exp(-iH_c\tau/\hbar)v_k = v_{k+1(\mathrm{mod}\ N)},$$

so that a clock will pass successively through the states $v_0, v_1, v_2, \ldots$ at time intervals τ.

Naturally, this discrete Hamiltonian and clock-time operator cannot satisfy $[T_c, H_c] = i\hbar$. A straightforward calculation gives

$$\langle u_m|[T_c,H_c]|u_n\rangle = \langle v_n|[T_c,H_c]|v_m\rangle = 0 \quad (n = m),$$
$$i\hbar\frac{2\pi i(n-m)/N}{1-\exp[2\pi i(n-m)/N]} (n \neq m).$$

This rather complicated result is due to the clock-time discontinuity when going from the $2j$th to the zeroth hour.

We now turn to examine the energetics of our clock. The clock-time eigenstates v_k satisfy $\langle H_c\rangle = 0$ and

$$(\Delta H_c)^2 = |H_c v_k|^2 = (\hbar^2\omega^2/N)\sum m^2 = \hbar^2\omega^2 j(j+1)/3.$$

We see that for large j, the energy uncertainty

$$\Delta H_c \simeq j\hbar\omega/\sqrt{3} \simeq (\pi/\sqrt{3})(\hbar/\tau)$$

is almost as large as the maximum available energy $j\hbar\omega$. Therefore our clock is an essentially nonclassical object. This will be reflected in its interaction with other objects, as shown below in this article.

In particular, it should be kept in mind that in our finite Hilbert space, no local function of θ [except $f(\theta) = \mathrm{const}$] is an operator. For example, consider $A = \cos\theta$ which satisfies

$$Au_m = (1/2)(u_{m-1} + u_{m+1}).$$

This is outside our Hilbert space if $m = \pm j$, therefore A is not an operator. Conversely, an operator with matrix elements

$$A_{mn} = (1/2)(\delta_{m,n+1} + \delta_{m,n-1}).$$

(with $|m| \leq j$ and $|n| \leq j$) corresponds to a nonlocal integration kernel in the θ representation. We shall therefore not be able to turn an interaction on and off at precise times, or even make it follow a smooth time evolution with few Fourier components. *Nonlocality in time* is apparently an essential feature of quantum clocks.

To conclude this section, we note that a more classical clock, with $\Delta H_c \ll \langle H_c\rangle_{\max}$, can be constructed by using a Hilbert space with many more states than actually needed for the required time resolution. For example, we can take

$$\psi = A\exp[M\cos(\theta-\theta_0)],$$

where M is a large number and A a normalization constant. The angular resolution is $\Delta\theta \simeq M^{-1/2}$ so that there are about $2\pi M^{1/2}$ distinct "pointer states." But a good representation of ψ by a Fourier series necessitates about $2M$ terms. The problem is not only that we are "wasting" most of our Hilbert space, but the energy uncertainty, with is of order $2M\hbar\omega \simeq 2M^{1/2}\hbar\omega/\Delta\theta$, is actually much higher than for a "minimal" clock with the same accuracy $\Delta\theta$. Therefore this "classical" clock will cause an even greater disturbance to the observed system.

III. TIME-OF-FLIGHT MEASUREMENT

Clocks are commonly used to measure the time needed for a particle to pass between two detectors. In principle, we could consider this pair of detectors as a quantum-mechanical scatterer and define a "delay time operator" like the one introduced by Jauch and Marchand.[3] However, our approach here is different: We want to describe an interaction with a real clock, having a Hamiltonian $H_c = \omega J$, coupled to the dynamical variables of the particle. Moreover, we want this clock to keep a permanent record of the time of flight, which can be read long after the process has been completed.

The simplest way of achieving this is to have the movement of the clock activated when the particle passes through the first detector and stopped when it passes through the second one. Let q_1 and q_2 be the positions of these detectors and define the projection operator

$$P(q) = \begin{cases} 1 & \text{if } q_1 < q < q_2, \\ 0 & \text{otherwise} \end{cases}.$$

Then a suitable Hamiltonian for the particle (mass m) and the clock is

$$H = (p^2/2m) + P(q)\omega J,$$

so that the clock runs only when the particle is between the detectors. Note that both H and J are constants of the motion.

The initial state of the clock is $v_0 = \sum u_n/\sqrt{N}$. However, it is easier to solve the equations of motion for the clock in an eigenstate of J, say u_n, and then to sum the solutions for all n, so as to get the solution corresponding to the initial state v_0.

If the state of the clock is u_n, the operator J can be replaced by the numerical constant $n\hbar$ and the Hamiltonian H simply represents a free particle with a square potential barrier of height $V = n\hbar\omega$ and length $L = q_2 - q_1$. Outside the barrier, the wave number is $k = (2mE)^{1/2}/\hbar$, where $E = p^2/2m$ is the constant value of H. Inside, it is $k' = [2m(E - V)]^{1/2}/\hbar$. Therefore the phase shift caused by the barrier is

$$(k' - k)L \simeq -n\omega L/(2E/m)^{1/2},$$

the right-hand side being a good approximation for $|V| \ll E$. Note that $v = (2E/m)^{1/2}$ is the classical velocity of the particle, so that L/v is the classical time of flight T. Then, if the incoming wave function (before the barrier) is

$$e^{ikq}v_0 = e^{ikq}\sum u_n/\sqrt{N},$$

the outgoing wave function (after the barrier) is

$$e^{ikq}\sum e^{-in\omega T}u_n/\sqrt{N} = e^{ikq}\sum e^{in(\theta-\omega T)}/(2\pi N)^{1/2}.$$

Therefore the pointer, which was initially directed toward

$\theta \simeq 0$, will be left oriented toward $\theta \simeq \omega T$. It will correctly indicate the time of flight between the detectors.

It is remarkable that this derivation never assumed that the particle was constrained to stay in a wave packet much smaller than L, or in general that it passed through each detector at a definite time. We wanted to measure only the time *interval* between the two detectors. The latter is perfectly well defined by the initial momentum $p = \hbar k$, even though the times of passage through each detector are completely uncertain.

Our only assumption was that $|V| \ll E$, so that the disturbance caused by the measurement is small. Now $|V|$ can be as large as $j\hbar\omega \simeq \pi\hbar/\tau$, therefore $\tau \gg \hbar/E$. This imposes a lower limit on the time resolution of the clock, which will in turn cause a limitation of the accuracy with which the particle velocity can be measured over a distance L:

$$\Delta v \simeq \tau v^2/L \gg \hbar/mL.$$

This result can also be written as $\Delta p \gg \hbar/L$, but it is *not* a Heisenberg uncertainty relation. It is an inherent limitation of the time-of-flight method to measure the velocity of a nonrelativistic particle.[4] It is of course quite possible to measure p with unlimited accuracy by a different method.

IV. WHEN DID THE ATOM DECAY?

The next example is related to the well-known story of Schrödinger's cat who is killed when a radioactive atom decays. As the Schrödinger equation causes the atom to be in a superposition of states (decayed and undecayed) the question is: When did Schrödinger's cat die?[5]

Let us replace Schrödinger's cat by a clock, which runs as long as the atom has not decayed. The question thus becomes: When did the clock stop, i.e., what is the location of its pointer when $t \to \infty$.

The Hamiltonian is

$$H = H_a + P_0 H_c,$$

where H_a is the Hamiltonian of the atom, P_0 is the projection operator for its original (undecayed) state, and $H_c = \omega J$.

For H_a, we shall use the following simple model, which gives an almost exponential decay[6]:

$$H_a = H_0 + V,$$

where H_0 has a continuous spectrum

$$H_0\phi(E) = E\phi(E), \qquad E > E_{\min}.$$

These eigenstates are normalized according to $\langle\phi(E')|\phi(E'')\rangle = \delta(E' - E'')$. Moreover, H_0 has one *discrete* eigenstate ϕ_0, with energy $E_0 > E_{\min}$, which is the initial state of the atom. This state is orthogonal to all the $\phi(E)$ and we assume that the only nonvanishing matrix elements of V are

$$\langle\phi_0|V|\phi(E)\rangle = \overline{\langle\phi(E)|V|\phi_0\rangle},$$

which shall be denoted as $V(E)$, for brevity. Furthermore, we assume that $V(E)$ is an almost constant function of E over a large domain on both sides of E_0 and then slowly falls off to zero for $|E - E_0| > W$. (Of course, $E_0 - W \geq E_{\min}$.)

The wave function of the atom can be written as

$$\psi = a_0\phi_0 \exp(-iE_0 t/\hbar) + \int a(E)\phi(E)\exp(-iEt/\hbar)dE,$$

where a_0 and $a(E)$ are functions of time. Initially, $a_0 = 1$ and $a(E) = 0$. The Schrödinger equation for the atom (without the clock) gives

$$i\hbar\dot{a}(E) = V(E)a_0 \exp[i(E - E_0)t/\hbar].$$

The Weisskopf–Wigner ansatz[7] for the survival amplitude is $a_0 = e^{-\gamma t/\hbar}$. Substitution in the preceding equation gives $a(E)$ explicitly. For $t \to \infty$, the result is

$$a(E) \to V(E)/(E - E_0 + i\gamma),$$

and ψ becomes

$$\psi(t) \to \int \frac{V(E)\phi(E)\exp(-iEt/\hbar)}{E - E_0 + i\gamma} dE.$$

Consistency ($\langle\psi|\psi\rangle = 1$) implies that

$$\int \frac{|V(E)|^2}{(E - E_0)^2 + \gamma^2} dE = 1,$$

and since we assumed that $V(E)$ is almost constant over a domain much larger than γ, we can write

$$[(E - E_0)^2 + \gamma^2]^{-1} \simeq (\pi/\gamma)\delta(E - E_0)$$

and obtain

$$\gamma = \pi|V(E_0)|^2,$$

which is Fermi's golden rule. [There is no "density of states" $\rho(E_0)$ here, because of the normalization chosen for $\phi(E)$.]

Let us now couple our atom to a clock:

$$H = H_a + P_0\omega J,$$

where P_0 is the projection operator on ϕ_0. As in Sec. III, the initial state of the clock is $v_0 = \Sigma u_n/\sqrt{N}$, but we shall first solve the problem for a clock in state u_n, so that J can be replaced by the numerical constant $n\hbar\omega$.

The only change in the Hamiltonian is that the energy of the initial state has been shifted from E_0 to $E_0 + n\hbar\omega$. Therefore the new value of γ will be $\pi|V(E_0 + n\hbar\omega)|^2$, which will be very close to the old value as long as

$$j\hbar\omega \frac{d}{dE_0}|V(E_0)|^2 \ll |V(E_0)|^2.$$

We again get a lower limit on the allowed time resolution:

$$\tau \gg \hbar\, d[\log|V(E_0)|^2]/dE_0.$$

A finer time resolution will alter the decay law. An extremely high time resolution ($j\hbar\omega \gg W$) will make, for most values of n, $|V(E + n\hbar\omega)|^2$ very small (or zero) and this will bring the decay process to a halt. This is the quantum analog of the famed Zeno paradox.[8]

In what follows, it will be assumed that we have kept the clock energy low enough, and therefore the time resolution poor enough, for the preceding inequality to be fulfilled, so that the decay law is not appreciably affected by the clock.

After a very long time, the combined state of the atom and the clock is given by

$$\psi = N^{-1/2}\sum u_n \int \frac{V(E)\phi(E)\exp(-iEt/\hbar)}{E - E_0 - n\hbar\omega + i\gamma} dE.$$

We are going to observe the clock, but not the decay products of the atom. Therefore the state of the clock (more exactly, the statistical properties of an ensemble of clocks, all of which have been subject to the same experiment) will be represented by the density matrix[9]

$$\rho = \mathrm{Tr}_a(|\psi\rangle\langle\psi|),$$

where Tr_a means that the trace must be taken on the atom degrees of freedom only. In other words, an expression $|u_n\phi(E)\rangle\langle u_m\phi(E')|$ will become

$$|u_n\rangle\langle u_m|\langle\phi(E')|\phi(E)\rangle = |u_n\rangle\langle u_m|\delta(E' - E).$$

We obtain

$$\rho = \frac{1}{N}\sum |u_n\rangle\langle u_m| \times \int \frac{|V(E)|^2 dE}{(E - E_0 - n\hbar\omega + i\gamma)(E - E_0 - m\hbar\omega - i\gamma)}.$$

Our assumptions on $|V(E)|^2$ allow us to replace it by γ/π. We can then safely extend the integration domain to $\pm\infty$ and obtain

$$\rho = \frac{1}{N}\sum \frac{|u_n\rangle\langle u_m|}{1 + i\alpha(n - m)},$$

where $\alpha = \hbar\omega/2\gamma$ is the angle through which the pointer of a classical clock would turn during an average atom lifetime $\hbar/2\gamma$. Note that ρ is Hermitian and that its trace is 1, as it should be (i.e., our approximations are consistent). However, $\rho^2 \neq \rho$, because after the atom degrees of freedom have been "traced out," the clocks are in a statistical mixture of states,[9] rather than a pure state like ψ.

We can now compute the probability $\langle P_k\rangle$ of finding a clock stopped at time

$$t_k = k\tau = 2\pi k/N\omega.$$

It is

$$\mathrm{Tr}(\rho P_k) = \frac{1}{N}\sum \frac{\langle v_k|u_n\rangle\langle u_m|v_k\rangle}{1 + i\alpha(n - m)}, = \frac{1}{N^2}\sum \frac{\exp[2\pi ik(n - m)/N]}{1 + i\alpha(n - m)}.$$

The double sum over n and m can be performed by first keeping $n - m$ fixed, and summing over $n + m$ (with the same parity as $n - m$ and within the limits $\pm 2j \mp |n - m|$). The result is

$$\langle P_k\rangle = N^{-2}\sum(N - |p|)e^{ip\theta}/(1 + i\alpha p),$$

where $p = n - m$ runs from $-2j$ to $2j$, and $\theta = 2\pi k/N$. This can also be written as

$$\langle P_k\rangle = \frac{1}{N}\left(\sum \frac{e^{ip\theta}}{1 + i\alpha p} - \frac{1}{N}\sum p \frac{e^{ip(\theta+\pi)}}{1 + i\alpha p}\right),$$

and it is easily shown that when N is large, the second term is negligible with respect to the first one (to prove this, the factor p can be replaced by $-i\partial/\partial\theta$). The first term can be recognized as the Fourier series expansion of

$$\frac{2\pi e^{-\theta/\alpha}}{N\alpha(1 - e^{-2\pi/\alpha})} = \frac{2\pi}{N\alpha}(e^{-\theta/\alpha} + e^{-(\theta+2\pi)/\alpha} + e^{-(\theta+4\pi)/\alpha} + \cdots),$$

where $0 \leqslant \theta < 2\pi$. (The factor $2\pi/N$ is the angle interval corresponding to the time resolution τ.)

We thus see that the clocks stop (the atoms decay, the cats die . . .) at times distributed according to the familiar exponential decay law, with due account taken of the fact that a clock may run through more than one cycle before stopping. Quantum theory is of course unable to predict when an *individual* atom will decay or an *individual* cat will die.[10]

(As a final note, it should be pointed out that the Hamiltonian $H = H_a + P_0 H_c$ is only one of many possibilities to observe the evolution of the unstable atom. Another, more general method of monitoring a nonstationary state would be described by the Hamiltonian

$$H = H_a + H_c + P_0 P_k H_b,$$

where H_b is the Hamiltonian of an auxiliary device, e.g., a second clock, and the other symbols have their usual meaning. Loosely speaking, the second clock runs only if $P_0 P_k = 1$, e.g., if the atom has not yet decayed when the first clock shows time t_k. Therefore the final recording of the second clock is proportional to the number of atoms surviving at time t_k, provided that no observation was made before or after t_k. The last remark is important when the decay law is not exponential, in particular when P_0 is an oscillating function of time.[6])

V. SPIN PRECESSION DURING PRESCRIBED TIME

A clock can be used not only to measure, but also to control the duration of a physical process. In this section, we shall switch a magnetic field on at a preset time t_a, to cause the precession of a spinning particle, and then switch it off at time t_b. The problem is to find the total precession angle.

The Hamiltonian, which is of course time independent, is

$$H = H_c + (P_a + P_{a+1} + \ldots + P_{b-1})H_s,$$

where H_c is the clock Hamiltonian as usual, P_k is the projection operator for clock-time t_k as defined in Sec. II, and H_s can be written as

$$H_s = \Omega\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

by choosing the spin quantization axis along the magnetic field. Ω denotes the precession angular velocity. The translational degrees of freedom of the spinning particle have been ignored.

The initial state of the clock and of the spinning particle is

$$\psi_0 = v_0\begin{pmatrix}1\\1\end{pmatrix}\Big/\sqrt{2}.$$

However, to discuss the equations of motion, it is preferable to first consider the case of spin aligned with the quantization axis, so that H_s becomes a numerical constant $\pm\hbar\Omega$. On the other hand, H_c is *not* a constant of the motion, because it does not commute with the P_k. We see that using a clock to control a process perturbs the clock mechanism!

We shall proceed under the assumption that $\Omega \ll \omega$, so that this problem can be treated by standard time-dependent perturbation theory. Taking the u_k as a basis, so as to make H_c diagonal, the Hamiltonian matrix is

$$H_{mn} = \hbar[\omega m\delta_{mn} \pm \Omega\langle u_m|(P_a + \ldots + P_{b-1})|u_n\rangle].$$

Now

$$\langle u_m|P_k|u_n\rangle = N^{-1}e^{2\pi ik(n-m)/N},$$

so that the perturbation term is a geometric series which can be summed as

$$P_{mn} = \begin{cases} \dfrac{e^{i(n-m)a\delta} - e^{i(n-m)b\delta}}{N(1 - e^{i(n-m)\delta})} & (m \neq n) \\ (b-a)/N & (m = n), \end{cases}$$

where $\delta = 2\pi/N$ is the small angle corresponding to the time resolution τ.

Writing the clock state as $\psi = \Sigma a_m(t)e^{-im\omega t}u_m$, the Schrödinger equation for the clock is

$$i\dot{a}_m = \pm\Omega\sum P_{mn}a_n e^{-i(n-m)\omega t}.$$

For example, if we have initially $a_n = \delta_{nk}$, we get to first order in Ωt,

$$i\dot{a}_k = \pm\Omega(b-a)/N,$$
$$i\dot{a}_m = \pm\Omega P_{mk}e^{-i(k-m)\omega t} \qquad (m \neq k),$$

whence

$$a_k = 1 \mp i\Omega t(b-a)/N,$$
$$a_m = \pm\Omega P_{mk}\frac{e^{-i(k-m)\omega t} - 1}{(k-m)\omega} \qquad (m \neq k).$$

The initial state

$$(2N)^{-1/2}\begin{pmatrix}1\\1\end{pmatrix}\sum u_k$$

thus becomes after a short time t (short enough so that Ωt is small, but not necessarily ωt):

$$\begin{aligned}\psi = (2N)^{-1/2}\Bigg\{&\begin{pmatrix}1\\0\end{pmatrix}\sum_k\Bigg[u_k e^{-ik\omega t}\left(1 - \frac{i\Omega t(b-a)}{N}\right) + \frac{\Omega}{\omega}\sum{}' u_m P_{mk}\frac{e^{-ik\omega t} - e^{-im\omega t}}{k-m}\Bigg] \\ &+ \begin{pmatrix}0\\1\end{pmatrix}\sum_k\Bigg[u_k e^{-ik\omega t}\left(1 + \frac{i\Omega t(b-a)}{N}\right) - \frac{\Omega}{\omega}\sum{}' u_m P_{mk}\frac{e^{-ik\omega t} - e^{-im\omega t}}{k-m}\Bigg]\Bigg\} \\ = (2N)^{-1/2}\sum_k u_k e^{-ik\omega t}&\Bigg\{\begin{pmatrix}1\\1\end{pmatrix} - i\Omega t\begin{pmatrix}1\\-1\end{pmatrix} \times\left[\frac{b-a}{N} + \sum{}' P_{km}\frac{e^{i(k-m)\omega t} - 1}{i(k-m)\omega t}\right]\Bigg\}.\end{aligned}$$

(Here Σ' means a sum over all $m \neq k$.) Our problem is to find the final state of the spinning particle, irrespectively of that of the clock. As in Sec. IV, we must replace ψ by a density operator $|\psi\rangle\langle\psi|$ and take the trace over the clock variables.

As a preliminary to this calculation, we note that

$$R(t) = \sum_k\left(\frac{b-a}{N} + \sum{}' P_{km}\frac{e^{i(k-m)\omega t} - 1}{i(k-m)\omega t}\right)$$

is real. Thus, neglecting as before terms of order Ω^2, we obtain

$$\rho = \frac{1}{2}\begin{pmatrix}1 & 1 - iR\Omega t/N \\ 1 + iR\Omega t/N & 1\end{pmatrix}.$$

This corresponds to a spin precession through an angle

$$\begin{aligned}\frac{R\Omega t}{N} &= \frac{\Omega t}{N^2}\sum_k\left(b - a + \sum{}'\frac{e^{i(m-k)a\delta} - e^{i(m-k)b\delta}}{1 - e^{i(m-k)\delta}} \times\frac{e^{-i(m-k)\omega t} - 1}{-i(m-k)\omega t}\right), \\ &= \frac{\Omega}{N^2}\int_0^t\sum_k\left(b - a + \sum{}'\frac{e^{i(m-k)a\delta} - e^{i(m-k)b\delta}}{1 - e^{i(m-k)\delta}}e^{-i(m-k)\omega t'}\right)dt'.\end{aligned}$$

It is easily seen that the $b - a$ term becomes negligible when N is large. As in Sec. IV, the double sum can be performed by summing first over $m + k$ (with result $N - |m-k|$) and then over $m - k$, which shall be called n. As before, the term including $|n|$ becomes negligible in the limit of large N, and we are left with

$$\frac{\Omega}{N}\int_0^t\sum_n\frac{e^{ina\delta} - e^{inb\delta}}{1 - e^{in\delta}}e^{-in\omega t'}dt' \simeq \Omega\int_0^t\sum\frac{e^{inb\delta} - e^{ina\delta}}{2\pi in}e^{-in\omega t'}dt'.$$

The right-hand side has been obtained here in the limit $\delta = 2\pi/N \ll 1$ (but $a\delta$ and $b\delta$ remaining finite). The integrand is easily recognized as the Fourier expansion of a square wave,

$$\sum\frac{e^{inb\delta} - e^{ina\delta}}{2\pi in}e^{-in\omega t'} = \begin{cases}1 & \text{if} \quad a\delta < \omega t' < b\delta, \\ 0 & \text{otherwise.}\end{cases}$$

Note that $a\delta/\omega$ and $b\delta/\omega$ are just the times t_a and t_b at which the magnetic field has been turned on and off.

Therefore the precession angle is given by the ramp function

$$\begin{array}{lll} 0 & \text{for} & t < t_a, \\ \Omega(t - t_a) & \text{for} & t_a < t < t_b, \\ \Omega(t_b - t_a) & \text{for} & t > t_b. \end{array}$$

This is of course the expected result. However, it was obtained here with a time-independent Hamiltonian. The only approximation we had to make were $j \gg 1$, which is reasonable for any good clock, and $j\omega \gg \Omega$, or $\tau \ll \Omega^{-1}$, i.e., the clock's time resolution is much finer than the precession period. This last condition, which is physically obvious, is needed so that in the Hamiltonian matrix H_{mn}, the perturbation ΩP_{mn} remains small compared to the maximum value of the main term $j\omega$. Otherwise the clock movement would be completely overwhelmed by its coupling to the device which it is supposed to control. (We also assumed that $\Omega t \ll 1$ or $\Omega \ll \omega$ in order to use *first-order* perturbation theory. In principle, an exact calculation could be done without this condition.)

VI. OUTLOOK

These results are disquieting (one could almost say self-defeating). Our aim was to describe in a "realistic" way the measurement of time, by including the clock mechanism in the Hamiltonian. We found that improving the time resolution increases the disturbance caused to the system under observation. Moreover, if we use a clock to *control* the evolution of a physical system, not only to *observe* it, the clock itself is perturbed.

Our calculations were based on the Schrödinger equation $i\hbar\dot{\psi} = H\psi$. But the mathematical definition of $\dot{\psi}$, namely,

$$\dot{\psi} = \lim_{\epsilon \to 0} [\psi(t + \epsilon) - \psi(t)]/\epsilon,$$

now appears operationally meaningless: ϵ cannot be smaller than the time resolution τ and the latter must remain finite, lest the behavior of the system or the clock is drastically altered. [We find here a curious analogy with numerical differentiation by a computer having a word of finite length: the limit $\epsilon \to 0$ cannot be implemented because as ϵ becomes smaller, the expression $\psi(t + \epsilon) - \psi(t)$, being the small difference of two large quantities, loses its accuracy and finally becomes meaningless.]

It thus seems that the Schrödinger wave function $\psi(t)$, with its continuous time evolution given by $i\hbar\dot{\psi} = H\psi$, is an idealization rooted in classical theory. It is operationally ill defined (except in the limiting case of stationary states) and should probably give way to a more complicated dynamical formalism, perhaps one nonlocal in time. Thus, in retrospect, the Hamiltonian approach to quantum physics carries the seeds of its own demise.

[a]On sabbatical leave from Technion—Israel Institute of Technology, Haifa, Israel.

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