Time in quantum mechanics: a story of confusion

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Abstract

Time and space play fundamentally different roles in quantum mechanics; this seems to be the received view on time in quantum mechanics. I have argued that the difference is an apparent one and that time and space need not and should not be given essentially different quantum mechanical treatments.

In this article I trace the genesis of the problem in the work of six of the founding fathers of modern quantum theory: Dirac, Heisenberg, Bohr, Schrödinger, von Neumann and Pauli, covering the period 1925–1933. Little coherence between the views of these authors has been found but, from the beginning, the mixing of notions from classical and quantum mechanics on the one hand, and from relativity theory on the other, has been an important source of confusion.

Keywords: History; Quantum mechanics; Time

1. Introduction

The problem of time in quantum mechanics has puzzled physicists right from the beginning and it is still an actively debated subject. Consider the following two quotations. The first is from von Neumann’s famous book:
First of all we must admit that this objection... points to an essential weakness which is, in fact, the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes the time \( t \) from the three space coordinates \( x, y, z \), and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those \( x, y, z \) closely connected with \( t \) by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter \( t \), just as in classical mechanics. (Von Neumann, 1932, p. 188; 1955, p. 354.)

The second quotation is from the Preface of a recent book on time in quantum mechanics:

*Time and quantum mechanics* have, each of them separately, captivated scientists and layman alike, as shown by the abundance of popular publications on “time” or on the many quantum mysteries or paradoxes. We too have been seduced by these two topics, and in particular by their combination. Indeed, the treatment of time in quantum mechanics is one of the important and challenging open questions in the foundations of quantum theory. (Muga, Sala Mayato, & Egusquiza, 2002.)

Roughly, the problem is that, while position operators are commonplace, no time operators occur in ordinary quantum mechanics. In view of relativity theory this seems problematic. Most present-day textbooks emphasize that space and time play fundamentally different roles in quantum mechanics.

I have argued that time poses no fundamental problem for quantum mechanics (Hilgevoord, 2002). If by space and time one understands the coordinates of a given space and time background, none of these coordinates are operators in quantum mechanics. If, on the other hand, one thinks of position and time as dynamical variables connected with specific physical systems situated in space–time, the representation of such variables by quantum mechanical operators is possible. However, neither position- nor time-operators are relativistically covariant concepts. The apparent problem seems to be caused by the concurrence of a number of factors of various kinds, which I will discuss presently.

In this article I will try to trace the origin of the problem in the work of six of the founding fathers of modern quantum mechanics: Dirac, Heisenberg, Bohr, Schrödinger, von Neumann and Pauli, covering the period from the beginning in 1925 to the famous footnote in Pauli’s 1933 Encyclopedia article. When I criticize some of their work of this period, I am fully aware, of course, that these were the heroic years when modern quantum mechanics was taking shape. I have confined myself to what was known at the time and have refrained from discussing important later developments.2

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2For recent review articles and references see Muga et al. (2002).
The article is organized as follows. The main part consists of six sections devoted to the work of each of the six physicists mentioned, where I have limited my attention to what they say about time in quantum mechanics. Reference to these sections is made by printing the name of the author in question in italics (Dirac, etc.). Preceding the main part is a brief résumé of classical Hamiltonian mechanics. In Appendix A explicit examples are treated to elucidate several passages of the main text. In the remainder of this introduction I discuss the factors which, I believe, have been instrumental in causing the (apparent) problem of time in quantum mechanics.

1.1. External, internal and general time

Leaving aside general relativity, physical systems are supposed to be situated in an independent, homogeneous and isotropic space and time, and physical theories, including quantum mechanics, are formulated relative to a given space–time background. I shall denote the coordinates of this external space and time by \(x, y, z, t\). In practice these coordinates are defined by various pieces of laboratory equipment, but these do not occur in the theory.

An important step was taken in special relativity theory, where the coordinates \(x, y, z, t\) became linked together in the Lorentz transformation.

Besides the external coordinates of space and time there are internal spatial and temporal variables connected with the specific physical systems the theory aims to describe, such as the position variables of particles and time variables of clocks. These variables are dynamical: they obey equations of motion. Whereas, in a given theory, there is only one external time parameter,\(^3\) there may be infinitely many internal ones. In general, the evolution of a physical system provides an internal time variable. For a suitable clock, however, the internal time must mimic the external time as closely as possible. In fact, the name ‘clock’ is reserved for those systems which fulfill this requirement to a high degree.

While the distinction between external and internal space and time variables occurs already in classical mechanics, in quantum mechanics this distinction assumes a fundamentally new significance: whereas the dynamical variables of physical systems become operators in quantum mechanics, the external space and time coordinates remain \(c\)-numbers. This may be seen as a lack of consistency of the present quantum mechanics, but it may also be taken as a fundamental aspect of it. The latter view, I believe, was taken by Bohr (cf. Bohr). In any case, the independent external \(c\)-number space–time background plays in quantum mechanics the same important role as it does in classical mechanics because its assumed symmetries provide the conservation laws of energy, momentum and angular momentum.

We conclude: if \(t\) denotes external time it is not an operator in quantum mechanics; if we look for time operators we must look for internal times provided by

\(^3\)Whittaker, in his classic treatise on analytical dynamics, makes this clear at the beginning of his book where he assigns, “once and for all”, a definite significance to the parameter \(t\) by assuming it to be determined by the rotation of the earth (Whittaker 1904, p. 27).
special physical systems, ‘clocks’. In Appendix A we show that the existence of ideal clocks is not inconsistent with quantum mechanics.

Quite often the word ‘time’ is used in an unspecified general sense (cf. Pauli). In such cases I shall use the phrase ‘general time’ or ‘time in general’.

1.2. Explicit time dependence

If a system is under the influence of time-dependent external forces its Hamiltonian depends explicitly on the external time parameter \( t \): \( H = H(q, p, t) \), where \( q \) and \( p \) denote the set of canonical variables of the system. Curiously, several authors\(^4\) take relativity theory to require, in this case, that \( t \) be incorporated in the set of canonical variables (cf. Dirac). However, the \( q \)'s and \( p \)'s are generalized variables: they may be angles, angular momenta, or even combinations of such variables, and there may be arbitrarily many of them. Therefore, I see no ground for the above requirement. Yet we will see how this illogical demand played a crucial role in creating the problem of time in quantum mechanics.

1.3. A confusing notation

While in classical particle mechanics the time coordinate \( t \) figures prominently, this is not so for the space coordinates \( x, y, z \). Quite often, a distinction between the coordinates of a point of space and of the position of a point particle in space is not even made. Only in field theories do the space coordinates occur explicitly alongside the time coordinate. Yet, as pointed out above, the conceptual difference between the coordinates of points of space and the position coordinates of point particles is evident: only the latter are dynamical variables. The mixing up of these two concepts is fostered by the frequent use of the notation \( x, y, z \) for both (cf. von Neumann). In the following I shall denote by \( x, y, z \) the Cartesian coordinates of a point of space, and by \( q_x, q_y, q_z \) the Cartesian position coordinates of a particle. The time coordinate \( t \) is the natural companion of the space coordinates \( x, y, z \), as is explicit in relativity theory. Considering \( t \) as the companion of the dynamical variables \( q_x, q_y, q_z \) has caused a lot of confusion.

1.4. The misleading case of a single particle

Connected with the minor role of the coordinates \( x, y, z \), in particle mechanics is the minor role of the total momentum \( P(q, p) \), the generator of translations in space, as compared to the prominent role of the total energy \( H(q, p) \), the generator of translations in time. Here, \( (q, p) \) denotes the set of canonical position variables and conjugate momenta of the particles. The reason is that the effect of a spatial translation on the dynamical variables of a system of particles is trivial and the real interest is in temporal translations, i.e. in the evolution of the system in time.

(see Hilgevoord, 2002). Again, in field theories $P$ and $H$ are both important and in relativistic theories they form the components of a four-vector.

In the case of a single particle, a case frequently considered in fundamental physics, $P(q, p)$ coincides with $p$, the canonical momentum of the particle, and the value of $H(q, p)$ is the energy $E$ of the particle. In relativity theory $E/c$ and $p$ form a four-vector $(E/c, p)$, and, since the only other variables appearing in the dynamical problem are $q$ and the time $t$, the suggestion that these also form a four-vector is very strong. It is salutary to examine the confusing situation occurring in this case. $H$, as the generator of evolution in time, is coupled with $t$, just as $P$ is coupled with $x$ (for simplicity we consider only one dimension). However, in the one particle case, $P = p$, and, as a canonical variable, $p$ is coupled with $q$. Although it might appear natural to consider the couples $(E, t)$ and $(p, q)$ as analogues, in fact, the true analogues are $(E, t)$ and $(P, x)$. The next step is to include the time parameter $t$ in the set of canonical variables as the conjugate of $E$. Again, several authors consider this to be a requirement of relativity! Contrary to this, I think that $t$ and the canonical variables are conceptually different quantities, and that they should be kept apart.

2. Classical Hamiltonian mechanics; the origin of the problem

Quantum mechanics was modeled on classical Hamiltonian mechanics (Dirac, 1925, 1926a). In this section the relevant elements of the classical formalism are briefly reviewed.

2.1. The Lagrange–Hamilton formalism

A system of $n$ degrees of freedom is described by a set of $n$ independent generalized coordinates $q_1(t), \ldots, q_n(t)$, functions of a time parameter $t$. The physical meaning of these dynamical variables may vary widely: they may be position variables of particles, angles describing the orientation of a rigid body, etc. The equations of motion of the variables are derived from the Lagrangian $L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ of the system, a function of the generalized coordinates and the corresponding generalized velocities $\dot{q}_1(t), \ldots, \dot{q}_n(t)$, where the dot denotes differentiation with respect to $t$. To go over to the Hamilton equations one introduces the so-called conjugate momenta $p_1(t), \ldots, p_n(t)$ of the generalized coordinates defined by $p_k = \frac{\partial L}{\partial \dot{q}_k}$. The Hamiltonian $H = H(q_1, \ldots, q_n, p_1, \ldots, p_n)$ is now defined as $H = \sum_k p_k \dot{q}_k - L$, where it is understood that the generalized velocities are expressed in terms of the momenta by inversion of the equations $p_k = \frac{\partial L}{\partial \dot{q}_k}$. The $2n$ independent variables $q_1, \ldots, q_n, p_1, \ldots, p_n$, are called the “canonical” variables; they define a point in a $2n$-dimensional space, the phase space of the system. We shall often use the abbreviated notation $q$ and $p$ for these sets of variables. We note again that these variables need not be particle positions and particle momenta; their physical meaning in various systems may be quite diverse. The time evolution of the
system is given by
\[ \frac{dA}{dt} = \{A, H\}, \]  
(1)
where \( A = A(q, p) \) is any arbitrary function of the canonical variables and the symbol on the right-hand side is the Poisson bracket, defined as
\[ \{A, B\} = \sum_{k=1}^{n} \left( \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right). \]  
(2)
Evidently, the Hamiltonian itself is a constant of the motion called the energy of the system. For the canonical variables the equation of motion (1) reduces to
\[ \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \]  
(3)
and the Poisson brackets of the canonical variables are
\[ \{q_k, p_l\} = \delta_{kl}, \quad \{q_k, q_l\} = \{p_k, p_l\} = 0, \]  
(4)
where \( \delta_{kl} \) is the Kronecker symbol. A canonical transformation is a transformation to a new set of variables that leaves the Hamiltonian equations invariant. The transition to quantum mechanics is made by representing the canonical variables by operators and by replacing the Poisson brackets by commutators according to the substitution \( \{,\} \to (\hbar)^{-1} [\,]. \) This is called canonical quantization.

2.2. Multiply periodic systems

A special class of dynamical systems of particular interest in the early days of quantum mechanics are the multiply periodic systems that were supposed to describe the motion of electrons inside the atom. The \( q \)'s and \( p \)'s are the position and momentum variables of the electrons, respectively. For such systems there exists a canonical transformation from the original variables to a new set of canonical variables \( w_1, \ldots, w_n, J_1, \ldots, J_n \), the so-called angle and action variables (called “uniformising variables” by Dirac), with the following properties:

i. They satisfy the relations
\[ \{w_k, J_l\} = \delta_{kl}, \quad \{w_k, w_l\} = \{J_k, J_l\} = 0. \]  
(5)

ii. The Hamiltonian is a function of the \( J \)'s only: \( H = H(J_1, \ldots, J_n) \).

iii. The original \( q \)'s and \( p \)'s are multiply periodic functions of the \( w \)'s of period \( 2\pi \).

The equations of motion of the action and angle variables are
\[ \frac{dJ_k}{dt} = \{J_k, H\} = 0, \quad \frac{dw_k}{dt} = \{w_k, H\} = \frac{\partial H}{\partial J_k} = v_k. \]  
(6)
The solution of these equations is very simple: \( J_k = \) constant, \( w_k = v_k t \) + \( \delta_k \), where the \( v_k \) depend on the \( J \)'s only and \( \delta_k \) are arbitrary constants. The constants \( v_k \) are the fundamental frequencies of the motion.
Note: The angle variables are also called phase variables or *phases*. Since quite distinct quantities are called phases in quantum mechanics, this name may cause—and indeed has caused—confusion (cf. Heisenberg).

2.3. Time-variables

The time dependence of the angle variables is so simple that they may function as clocks. The variable \( \tau_i \equiv w_i/v_i \) has the simple equation of motion

\[
\frac{d\tau_i}{dt} = \{\tau_i, H\} = 1.
\]

We shall call such a variable a time-variable. If the frequency \( v_i \) of the angle variable \( w_i \) is independent of the \( J \)'s then \( \tau_i \equiv w_i/v_i \) and \( \eta_i \equiv v_i J_i \) form a canonically conjugate pair of variables.

Note the following points:

(a) Systems having time-variables are special; they are ‘clocks’.
(b) A system may have many time-variables \( \tau_i \) with associated conjugate momenta \( \eta_i \).
(c) By definition the \( \tau_i \) satisfy Eq. (7); in particular \( \{\tau_i, H\} = 1 \).

The case of a single time-variable is especially misleading since in this case \( H \) coincides with the canonical conjugate \( \eta \) of \( \tau \) and we have \( \{\tau, H\} = \{\tau, \eta\} = 1 \) (cf. Appendix A).

2.4. Explicit time dependence

Up to this point the system was supposed to be isolated, implying that \( H \) does not explicitly depend on \( t \) and is a conserved quantity. There are important applications where the system is subjected to time-dependent *external* forces; the Hamiltonian, then, depends explicitly on \( t \): \( H = H(q, p, t) \). Instead of (1) one has

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\},
\]

for the time evolution of a function \( A = A(q, p, t) \). In particular, \( dH/dt = \partial H/\partial t \), so \( H \) is not conserved.

2.5. Time as a canonical variable

Notwithstanding the conceptual difference between the dynamical variables and the evolution parameter, one sometimes finds it useful to include \( t \) in the set of dynamical variables of a system. This can be done by introducing a new evolution parameter \( u \) of which all dynamical variables, including \( t \), are supposed to be functions. To preserve the original Lagrangian equations of motion one must replace the Lagrangian \( L \) by a new Lagrangian \( A = t'L(q, q'/t', t) \) where the prime denotes

\footnote{See Goldstein (1980) and the other references in footnote 4.}
differentiation with respect to $u$. By definition, the momentum conjugate to $t$ is $p_t = \partial L / \partial \dot{t}$. Using $\dot{q} = q / t'$, one finds

$$p_t = L - t' \frac{\partial L}{\partial \dot{t}'},$$

that is, $p_t$ equals minus the original Hamiltonian: $p_t = -H(q_1, \ldots, q_n; t, p_1, \ldots, p_n)$. This result is true whether or not the original Lagrangian depends explicitly on $t$. Several authors have claimed that, in the first case, relativity theory requires the incorporation of the (external) time parameter $t$ in the set of dynamical variables (cf. Dirac, and footnote 4). In Section 1.2 I have argued that there is no basis for this claim. Nonetheless, it is often said to be a ‘well-known’ fact that time and energy (conveniently forgetting about the minus sign) form a canonical pair in classical mechanics (cf. Heisenberg, von Neumann). Time should here be taken in a general sense, not related to a specific physical system. By the canonical quantization procedure ‘time’ should then become an operator in quantum mechanics and, since this is not the case, we have arrived at the origin of the problem of time in quantum mechanics.

However, transforming the external time parameter into a canonical variable is not a natural thing to do. First, note again that it is minus the (original) Hamiltonian which is conjugated to $t$, not $H$ itself as relativity theory would seem to suggest. Second, the required inversion of the equations $p_t = \partial L / \partial \dot{t}', p_k = \partial L / \partial q_k'$, turns out to be impossible. This follows from the identity

$$p_t = -H(q_1, \ldots, q_n; t, p_1, \ldots, p_n),$$

through which the new canonical variables become mutually dependent. Thus, the usual road from the Lagrange formalism to the Hamilton formalism is blocked.

At this point, one might well forget about a Lagrangian and start from a Hamiltonian right away. However, if one is to preserve the original equations of motion one now finds that relation (8) reappears as a constraint (not an identity) on the extended set of canonical variables. Thus we get a theory with a constraint. This is a quite serious departure from the ordinary Hamiltonian scheme that leads to problems, as we shall see (cf. Dirac). Therefore, the statement that the time parameter can be looked upon as just another canonical variable is a bit rash, to say the least.

Furthermore, the idea that it is relativity theory that requires the time parameter to be included in the set of dynamical variables to join the ‘space’ coordinates is even more far-fetched. The $q$’s are generalized coordinates; they need not have the meaning of positions. But, even if the $q$’s are position coordinates of particles there is only one time parameter $t$ available to join $n$ position variables. The only feasible example is that of one-particle, and it is indeed to this special case that almost all treatments of covariant Hamiltonian mechanics confine the discussion. In fact, a relativistic Hamiltonian framework for a system of interacting particles does not seem to be possible (Goldstein, 1980, Chap. 8-4). We shall see that Dirac was quite aware of the very limited scope of relativistic classical particle mechanics.

The consideration of time as a canonical variable played a major role in Dirac’s early efforts to reconcile quantum mechanics and relativity theory and, although this
We conclude that in Hamiltonian mechanics \( t \) should keep its role of a time parameter. There is no fundamental reason to release \( t \) from this role and promote it to the status of a dynamical variable. From a relativistic point of view the true companions of the time parameter \( t \) are the space coordinates \( x, y, z \). These are the coordinates appearing in a Lorentz transformation. None of them is an operator in quantum mechanics.

3. Quantum mechanics 1925–1933

In this section I will trace the problem of time in quantum mechanics through the work of Dirac, Heisenberg, Schrödinger, Bohr, von Neumann and Pauli during the period 1925–1933. Of these authors, Dirac was the first to deal intensively with this problem.

3.1. Dirac

Early in September 1925, Paul Adrien Maurice Dirac (1902–1984) had seen the proofs of Heisenberg’s famous paper marking the beginning of modern quantum theory (Heisenberg, 1925). Being convinced of the indispensability of Hamiltonian mechanics in atomic theory, he was not satisfied with Heisenberg’s exposition and tried to adapt it to the Hamiltonian formalism. In a short time he developed a formalism by which quantum mechanics could be formulated in close analogy to Hamiltonian mechanics (Jammer, 1989, p. 236). Indeed, the desire to bring quantum mechanics into Hamiltonian form was to preoccupy him his whole life long (Dirac, 1964). The subject of time appears for the first time in his article “Relativity quantum mechanics with an application to Compton scattering” (Dirac, 1926a), where he tried to extend Heisenberg’s quantum mechanics into a relativistic theory. For our purpose, we need only discuss Section 2 of that paper, entitled “Quantum time”. (The notation will occasionally be adapted.) As his starting point Dirac takes the canonical formalism described in the preceding section, explicitly connecting the case of a time-dependent Hamiltonian and the theory of relativity (cf. Section 1.2):

It will be observed that the notion of canonical variables plays a very fundamental part in the theory. Any attempt to extend the domain of the present quantum mechanics must be preceded by the introduction of canonical variables into the corresponding classical theory, with a reformulation of this classical theory with Poisson Brackets instead of differential coefficients. The object of the present paper is to obtain in this way the extension of the quantum mechanics to systems for which the Hamiltonian involves the time explicitly and to relativity mechanics. (Dirac, 1926a, p. 406.)

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6E.g. Palmer and Rogalski (1999, p. 188).
and

Consider a dynamical system of $n$ degrees of freedom for which the Hamiltonian $H$ involves the time explicitly. The principle of relativity demands that the time shall be treated on the same footing as the other variables, and so it must therefore be a $q$-number. On the classical theory it is known that one may solve the problem by considering the time $t$ to be an extra co-ordinate of the system, with minus the energy (or perhaps a slightly different quantity) $W$ as conjugate momentum. (Dirac 1926a, p. 407.)

Accordingly, two variables, $t$ and $-W$, are added to the original $n$ variables where $-W$ is the conjugate momentum of $t$ (denoted $p_t$ in Section 2.5). Eq. (8) now appears in the form

$$H(q_k, p_k, t) - W = 0 \quad (k = 1, \ldots, n).$$

(9)

In quantum mechanics, $t$ and $-W$ become operators ($q$-numbers) satisfying the usual canonical commutation relations, i.e. they commute with the other canonical variables and among themselves satisfy the relation

$$tW - Wt = -i\hbar.$$

(10)

Now, Eq. (9) leads to a difficulty. Since $H$ does not depend on $W$, $t$ commutes with $H$, so that Eqs. (9) and (10) are inconsistent. Dirac notes this difficulty but has no ready answer. He only remarks that the problem does not give any trouble in his paper since it deals with a single simple system: “as we shall follow the classical theory so closely that it will be immediately obvious whether any quantum operation corresponds to a legitimate classical operation or not”. Next, he notes that to solve the quantum problem one must determine a set of ‘uniformising’ variables $J_0, \ldots, J_n, w_0, \ldots, w_n$, as in the classical theory, where one of the $w$’s, say $w_0$, must be the time, $w_0 = t$. These variables must satisfy the canonical commutation relations

$$[w_u, w_v] = [J_u, J_v] = 0, \quad [w_u, J_v] = i\hbar \delta_{uv} \quad (u, v = 0, 1, \ldots, n).$$

(11)

The Hamiltonian Equation (9) must become a relation between the $J$’s only and the original variables must be multiply periodic functions of the $w_k (k = 1, \ldots, n)$. In particular, we have $[t, J_0] = i\hbar$, so, in view of (10), $J_0$ is essentially $-W$ and the Hamiltonian Equation (9) takes the form $H_0 + J_0 = 0$, where $H_0$ is a function of $J_1, \ldots, J_n$ only.

Less than three months later, Dirac abandons the method described above, which he then calls “rather artificial”, and turns to Schrödinger’s wave mechanics. In his article “On the theory of quantum mechanics” (Dirac, 1926b) he starts from the Schrödinger equation

$$\{H(q_k, i\hbar \partial / \partial q_k) - W\} \Psi = 0,$$

(12)

which he sees as closely connected to the Hamiltonian equation

$$H(q_k, p_k) - W = 0.$$

(13)
In the present paper, Schrödinger’s theory is considered from a slightly more general point of view, in which the time \( t \) and its conjugate momentum \(-W\) are treated from the beginning on the same footing as the other variables. (Dirac, 1926b, p. 662.)

Whereas in the previous papers the nature of the dynamical variables was left unspecified (only their algebraic properties being given), the \( q \)'s and \( t \) are now counted as ordinary real numbers and the \( p \)'s and \(-W\) are taken to be the differential operators

\[
p_k = -i\hbar \partial /\partial q_k, \quad -W = -i\hbar \partial /\partial t. \tag{14}
\]

I will not discuss this article any further but only remark that the time derivative is considered as an operator on the same footing as the \(-W\), and that an explicit time dependence of the Hamiltonian, considered so important in the previous paper, is not mentioned any more.

Three months later the theory is applied to a single electron exposed to electromagnetic radiation (Dirac, 1927a; received 8 November 1926). The Hamiltonian equation of a free relativistic electron is taken to be

\[
m^2c^2 - W^2/c^2 + p_1^2 + p_2^2 + p_3^2 = 0.
\]

Schrödinger’s wave equation becomes

\[
[m^2c^2 - W^2/c^2 + p_1^2 + p_2^2 + p_3^2]\Psi = 0, \tag{15}
\]

where the symbols \( p_1, p_2, p_3 \) and \(-W\) mean the operators

\[-i\hbar \partial /\partial x_1, \quad -i\hbar \partial /\partial x_2, \quad -i\hbar \partial /\partial x_3 \quad \text{and} \quad -i\hbar \partial /\partial t.\]

We note the change from \( q \) to \( x \) in Dirac’s notation for the position variables (cf. (14)). This change is not completely innocent. Whereas in the beginning \( t \), as a parameter, stood apart from the dynamical variables \( q_k \), to which it was somewhat artificially added only later, now the notation strongly suggests a natural equivalence of the position variables \( x_1, x_2, x_3 \) and the time parameter \( t \) as dynamical variables. We shall see that Dirac, for the most part, resisted this suggestion but many others fell victim to it (cf. von Neumann). Clearly, the confusion is caused by the clash of two formalisms, the canonical formalism of classical mechanics with its \( q \)'s and \( p \)'s on the one hand, and relativity theory with space–time coordinates \( x_1, x_2, x_3, t \) on the other.

In his next article “The physical interpretation of the quantum dynamics” (Dirac, 1927b; received 2 December 1926), Dirac seems to have abandoned the idea of time as a dynamical variable. The article, which deals with the general, non-relativistic theory, begins thus:

The new quantum mechanics consists of a scheme of equations which are very closely analogous to the equations of classical mechanics, with the fundamental difference that the dynamical variables do not obey the commutative law of multiplication, but satisfy instead the well-known quantum conditions. It follows that one cannot suppose the dynamical variables to be ordinary numbers
(c-numbers), but may call them numbers of a special type (q-numbers). The theory shows that these q-numbers can in general be represented by matrices whose elements are c-numbers (functions of a time parameter). (Dirac, 1927b, p. 621.)

The discussion of a relativistic wave equation is resumed in the articles on “The quantum theory of the electron” in which the wave equation (15) is replaced by one that is linear in the derivatives, the famous relativistic Dirac equation of the electron (Dirac, 1928a,b). In this equation the wave function \( \psi(x_1, x_2, x_3, t) \) is a four-component spinor, describing a particle with spin \( \frac{1}{2} \). We need not discuss the original articles but may switch directly to Dirac’s great book “Quantum Mechanics”, the first edition of which appeared in 1930 (Dirac, 1930). We will be mainly concerned with this first edition (the second edition appeared in 1935). In the first six chapters of the book Dirac develops his well-known symbolic algebra of states and observables. The formalism is explicitly non-relativistic: observables, denoted \( \xi_k \), are taken to represent dynamical variables at a given time \( t \) (§ 9). The time \( t \) is considered as a simple parameter, a c-number, throughout the book. The next six chapters are devoted to applications. Only in the thirteenth and last chapter is relativistic quantum mechanics discussed, particularly the Dirac equation of the electron.

Before turning to the last chapter of the book we must mention sections 36 and 37 of Chapter VI. In these sections the effect is considered of displacements \( \delta x \) and \( \delta t \) of the system in ordinary space and time. Dirac shows that the effect on the states and observables is given by linear operators \( d_x \) and \( d_t \), the generators of infinitesimal displacements in space and time, which he calls the space-displacement operator and the time-displacement operator, respectively. Next, a special case is considered where the \( q_k \) are the Cartesian coordinates \( x, y \) and \( z \) of a single particle. The conjugates \( p_x, p_y \) and \( p_z \) of these observables are the components of the momentum of the system. For simplicity, we consider one dimension \( x \). Dirac is able to show that in the \( x \)-representation, in which a state \( \psi \) is given by the wave function \( (x|\psi) \), \( d_x = -\partial/\partial x \) and \( d_x x - xd_x = -1 \). From this and the canonical commutation relations one derives \( p_x = i\hbar d_x \). Similar results hold for \( y \) and \( z \). Next, Dirac considers the action of the time-displacement operator on the state \( \psi \). He shows \( d_t = -\partial/\partial t \), where \( t \) is the time evolution parameter. Everything seems to work out as for space except for a change of sign: to get the correct equation of motion one must put \( H = -i\hbar d_t \), where \( H \) is the Hamiltonian. Dirac notes the difference in sign as compared to \( p_x = i\hbar d_x \) but gives no explanation. The difference is caused by the fact that while \( x \) is an observable here, \( t \) is only a parameter: there is no analogue for time of the equation \( d_x x - xd_x = -1 \). Interestingly enough, the time-displacement operator has disappeared from the second and later editions of the book. However, the unfortunate hybrid notation \( x, y, z, t \), where the first three symbols denote position observables (or eigenvalues of them) and the last symbol is a c-number, has continued to cause confusion.

We now turn to the last chapter of the book: “Relativity Theory of the Electron”. Its first section (§73), “Relativity treatment of a single particle”, is of particular interest. Here, Dirac expresses the opinion that the general theory of states and observables given in the first part of the book will apply also to relativistic treatments of dynamical systems. However, because observables \( \xi_t \) are taken to represent
dynamical variables at a given time, it will be very complicated to build up a relativistic theory using these quantities and they are not expected to play any fundamental role in a relativistic theory. He concludes:

A possible way out of the difficulty would be to build up a purely field theory and to take as observables the values of the field quantities at definite points in space–time. This appears to be the most straightforward way of dealing with general dynamical systems on relativity lines, but it involves complicated mathematics and appears to be too difficult for practical application. (Dirac, 1930, p. 238.)

Lacking a general scheme of relativistic quantum mechanics, all one can do is to solve special problems in a Lorentz-invariant way:

The difficulty of a relativity treatment becomes much less severe when one confines one’s attention to the problem of a single particle moving in a given field of force. If we now take a representation in which the observables \( x_t, y_t, z_t \) specifying the position of the particle at time \( t \) are diagonal, we have as the wave function representing a state a function \( (x_t, y_t, z_t) \) of the three variables \( x_t, y_t, z_t \) depending on the parameter \( t \), which is the same as a function \( (x y z t) \) of the four variables \( x, y, z, t \). The domain of our wave function thus becomes identical with the ordinary space–time continuum, and this circumstance makes possible an elementary treatment of the problem and allows us to use considerations which cannot be extended to more general dynamical systems. (Dirac, 1930, p. 238.)

The forced restriction to a simple system should not be regarded as a defect of the quantum theory, since it is in perfect analogy with the classical theory. Relativistic classical mechanics does not involve any such general scheme as contact transformation theory of non-relativistic classical mechanics, but consists in the solution of comparatively special problems (Dirac, 1935, p. 251.)

Dirac goes on to introduce the momentum and energy of the particle for which, after some hesitation, he takes the operators \( p_x = -i\hbar \partial / \partial x, p_y = -i\hbar \partial / \partial y, p_z = -i\hbar \partial / \partial z \) and the corresponding \( W = i\hbar \partial / \partial t \), in agreement with sections 36 and 37. Remarkably, in the second edition of his book (Dirac, 1935, §68), it is asserted (again without explanation) that the sign difference between the first three and the last of these operators, which in his earlier papers and in the first edition of his book arose in a purely non-relativistic context, is required by relativity!

In this way, deliberately mixing quantities of a different kind, Dirac derives his famous relativistic wave equation for a four-component spinor wave function \( \psi(x, y, z, t) \), whose most important feature, of course, is that it describes a particle with spin. The rest of the chapter is devoted to a discussion of the consequences of this, and to the interpretation of the negative energy solutions of the wave equation. More interesting for us is the interpretation of the variables \( x, y, z \). Dirac turns to this problem only in the second and later editions of his book, where he shows that the eigenvalues of the velocity components \( dx/dt, dy/dt, dz/dt \) are \( \pm c \), a result first
derived by Breit in 1928. This surprising outcome points to an altogether different relation between velocity and momentum from what one has in classical relativistic mechanics. Moreover, the components of the velocity do not commute. Dirac tried to neutralize the impact of these results by referring to the ways such velocities are actually measured. However, in 1949, Newton and Wigner, studying the conditions that an acceptable position operator must satisfy, showed that the actual position operator of the electron is an essentially non-relativistic quantity, different from $x, y, z$. (Newton & Wigner, 1949. Also, Foldy & Wouthuysen, 1950.) In fact, a relativistically covariant position operator of a particle does not exist (Wightman, 1962). In the meantime, however, the interpretation of the relativistic Dirac equation had shifted from a single particle wave equation to a field equation for an operator field $\psi(x, y, z, t)$, where the $x, y, z, t$ are now taken to be simple $c$-numbers, the coordinates of a point of spacetime, and the problem of time disappeared from the formalism.7

(1) To obtain a relativistic generalization of Heisenberg’s quantum mechanics, Dirac introduces the time $t$ and minus the energy $W$ as canonically conjugate variables and adds the commutation relation $tW - Wt = -i\hbar$ to the relations $qp - pq = i\hbar$. This leads to an inconsistency because of the Hamiltonian equation $H(p, q, t) - W = 0$.

(2) Dirac turns to the Schrödinger theory and puts $-W = -i\hbar\partial/\partial t$ as a differential operator on a par with $p_k = -i\hbar\partial/\partial q_k$.

(3) In his book ‘Quantum Mechanics’ Dirac develops his symbolic algebra of observables and states. Since observables are taken to represent dynamical variables at a given time the theory is non-relativistic. The step to a relativistic wave equation can only be taken in the case of a single particle. To facilitate this step, the original notation $x_t, y_t, z_t$ for the electron position observables at parameter time $t$ is changed into $x, y, z, t$ and a relativistic wave equation is constructed from the operators $p_x = -i\hbar\partial/\partial x, p_y = -i\hbar\partial/\partial y, p_z = -i\hbar\partial/\partial z, W = i\hbar\partial/\partial t$. A problem arises because of the presence of negative energy solutions of the wave equation. Also, the interpretation of the $x, y, z$ symbols as particle position operators leads to problems.

(4) It becomes clear that the Dirac spinor wave function should not be interpreted as the wave function of a single particle but rather as an operator field in spacetime acting on a Hilbert space with arbitrary particle number. The symbols $x, y, z, t$ are now interpreted as the $c$-number coordinates of a point of spacetime. A time observable has disappeared from the formalism.

3.2. Heisenberg

Heisenberg’s (1925) article “Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen” is regarded as the beginning of modern quantum mechanics. In this article, Werner Heisenberg (1901–1976) introduced the transition quantities and their peculiar multiplication rule that Max Born recognized as being
the rule of matrix multiplication. Within two months, Heisenberg’s results were generalized by Born and Jordan and put in the form of the famous commutation relation $pq - qp = -i\hbar$ (Born and Jordan, 1925). One month later the three men published an exhaustive exposition of the new matrix mechanics, the so-called ‘Drei-Männer-Arbeit’ (‘three-men-paper’), (Born, Heisenberg, & Jordan, 1926). What is said about time in this article? In Kap. 1 §5 systems are considered for which the time enters explicitly into the Hamiltonian and the method, explained in Section 2.5 of the present paper, by which the time can be treated as a canonical variable is described. However, no commutation relation for a ‘time-matrix’ is given. Rather, time figures as an ordinary parameter in the time-dependent part of the Hamiltonian. This part is treated as a perturbation to the main, time-independent part of the Hamiltonian. Dirac, who, only a few months later, considered $t$ as a $q$-number satisfying canonical commutation relations, comments on this as follows: ‘In consequence of…the fact that $t$ commutes with each of the $p$’s and $q$’s, Born, Heisenberg and Jordan’s perturbation theory for systems for which the Hamiltonian contains the time explicitly, in which $t$ is treated as a $c$-number, can be justified.’ (Dirac, 1926a, p. 409). One month after finishing work on the three-men-paper Heisenberg wrote a review article of this work (Heisenberg, 1926). Again, time is treated as a simple classical parameter. But in the last paragraph of this paper Heisenberg mentions the problem of time when he remarks that the theory must still be considered to be incomplete:

Eine erhebliche Schwierigkeit liegt besonders darin, dass die Zeit in der Theorie scheinbar eine andere Rolle spielt und formal anders behandelt wird als die räumliche Koordinaten. (Heisenberg, 1926, p. 705.)

[A considerable difficulty is in particular that time seems to play a different role in the theory, and it is formally treated differently from the spatial coordinates. (Translation J.H.).]

From the context of this quotation it is evident that the term ‘spatial coordinates’ refers to particle position variables. We note that while the article deals with general dynamical variables, including angle and action variables, the time, in the above quotation, is specifically compared to ‘spatial coordinates’. Heisenberg’s wish to treat time in the same way as the particle position variables and his use of the term ‘spatial coordinates’ for the latter, suggests a confusion between particle positions and space coordinates.

For our purpose the most important of Heisenberg’s papers is the one in which he introduced the uncertainty relations (Heisenberg, 1927, received 23 March). [Most of our quotations will be taken from the English translation in Wheeler and Zurek (1983)]. The purpose of the paper was to make visualizable (‘anschaulich’) the physical meaning of the commutation relations. The article shows the signs of being hastily written; Heisenberg was anxious to publicize his new ideas and was unwilling to have his paper first scrutinized by Bohr. A rudiment of their disagreement is the ‘Addition in proof’ at the end of the article. A nice account of this episode is given by Rosenfeld (Wheeler and Zurek, 1983, p. 57).
The title of the first section of the article is: “Concepts: position, path, velocity, energy.” Time is conspicuously missing from this list in spite of the fact that time plays a considerable role in this section and in the rest of the paper. Perhaps Heisenberg was not too sure about the status of time in quantum mechanics. After having discussed the commutation relation \( pq - qp = -i\hbar \) and the associated uncertainty relation \( p_1 q_1 \sim \hbar \) for the position and momentum of a particle, Heisenberg turns to other quantities, notably the phase of the motion of an electron in an atom. The following passage is crucial to the history of the problem of time in quantum mechanics:

The distinction between classical and quantum mechanics consists rather in this: classically we can always think of the phase as determined through suitable experiments. In reality, however, this is impossible, because every experiment for the determination of phase perturbs or changes the atom. In a definite stationary “state” of the atom, the phases are in principle indeterminate, as one can see as a direct consequence of the familiar equations

\[
Et - tE = -i\hbar \quad \text{or} \quad Jw - wJ = -i\hbar,
\]

(16)

where \( J \) is the action variable and \( w \) is the angle variable. (Heisenberg, 1927, p. 177; Wheeler and Zurek, 1983, p. 66.)

This paragraph and the discussion following it, embody, I believe, much of the confusion about time in quantum mechanics. The passage raises immediate questions. How ‘familiar’ is the first equation in (16) and what, exactly, does it mean? The only publication prior to Heisenberg’s article I know of, containing something like the first Eq. (16), is Dirac’s article on relativity quantum mechanics (Dirac, 1926a; our Eq. (10)). In that article \( t \) is the (originally external) time parameter, turned into an internal variable by the procedure described there. But note the difference in sign between Dirac’s and Heisenberg’s versions. The sign in (10) is a consequence of Dirac considering, correctly, minus the energy as the canonical conjugate of time. Hence, if we interpret the first Eq. (16) as a canonical commutation relation for the general time observable, it has the wrong sign. In a paper with Born (Born and Heisenberg, 1928, p. 160) written in October 1927, the authors explicitly say that the energy \( W \) and the time \( t \) (meaning the external time parameter) form a canonical pair and there also they give the incorrect equation \( Wt - tW = -i\hbar \). This would suggest that \( t \) in (16) must be interpreted as the general time observable and that the sign error is just a slip (though not a completely innocent one, as we will see). However, there is another possibility. What exactly is the meaning of the word “or” in (16)? Does it indicate two essentially different things, or should it be taken to indicate two alternative ways of expressing the same thing? In the latter case the symbol \( t \) should be interpreted as a quantity closely related to the phase angles \( w \), i.e. as an internal time-variable like the one in (7). In that case, the sign in (16) would be correct. The quoted passage, and much of the rest of Heisenberg’s article, point to this latter alternative: sometimes \( E \) and \( J \) are simply identified and sometimes \( t \) is actually called a phase. However, Heisenberg does not define the various dynamical variables as operators (or matrices); he rather illustrates
their meaning in various experimental situations leaving much room for ambiguity. His first example of Eq. (16) concerns the measurement of the energy of electrons in rectilinear motion:

This measurement in principle can be carried out with arbitrary accuracy if only one foregoes the simultaneous determination of the position of the electron or its phase (see the determination of \( p \), above), corresponding to the relation

\[
Et - tE = -i\hbar. \quad (\text{Heisenberg, 1927, p. 178; Wheeler and Zurek, 1983, p. 67.})
\]

This seems clear enough: \( t \) is a phase. But what is the phase of an electron in rectilinear motion? The text continues:

The Stern–Gerlach experiment allows one to determine the magnetic or an average electric moment of the atom, and therefore to measure quantities which depend only on the action variable \( J \). The phases remain undetermined in principle. It makes as little sense to speak of the frequency of a light wave at a definite instant as of the energy of an atom at a definite moment. Correspondingly, in the Stern–Gerlach experiment the accuracy of the energy measurement decreases as we shorten the time during which the atom is under the influence of the deflecting field. (Heisenberg, 1927, p. 178; Wheeler and Zurek, 1983, p. 67.)

By comparing the deflection of the beam in the Stern–Gerlach magnet to the natural broadening of the beam, Heisenberg derives the relation

\[
E_1 t_1 \sim h, \quad (17)
\]

where \( E_1 \) is the precision of the energy measurement by the Stern–Gerlach apparatus and \( t_1 \) the time during which the atom is under the influence of the deflecting field. He concludes:

This equation corresponds to Eq. (16) and shows how a precise determination of energy can only be obtained at the cost of a corresponding uncertainty in the time. (Heisenberg, 1927, p. 179; Wheeler and Zurek, 1983, p. 68.)

This sentence concludes the first section of the article. What can we make of it? To an unsuspecting reader the most obvious reading is that ‘time’ here refers to an external time, a parameter of the experimental set-up. The experimenter can change this time at will, rendering the energy measurement more accurate or less accurate, but there is no uncertainty in this time. That reading, however, would make the time a \( c \)-number and the example, as an illustration of (16), would make no sense. Yet, precisely this reading of the uncertainty principle for energy and time, namely that an accurate energy measurement needs a long (external) time, has gained popularity in the older literature (cf. von Neumann, Heitler, 1954, p. 184), and it can still be found in more recent publications. This interpretation has been shown wrong, however, and in present-day quantum mechanics it is assumed that all observables can be measured with arbitrary accuracy in an arbitrarily short (external) time and the energy is no exception to this. Moreover, if (16) is to be considered the analogue of

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the relation \( pq - qp = -i\hbar I \), the operators in (16) should represent dynamical variables at a given instant of time.

In his book, based on lectures given in the spring of 1929, the Stern–Gerlach experiment is discussed once again, this time in more detail (Heisenberg, 1930, Chap. II, 2d). The purpose of the experiment is to separate atoms in the stationary states \( E_n \) and \( E_m \), respectively. As before, a time \( T \) is needed in order that the separation of the beams by the deflecting force exceeds the natural broadening of the beam. (This broadening, by the way, can be seen as a consequence of the uncertainty relation between position and momentum in the direction perpendicular to the beam.) The condition on \( T \) comes down to \( |E_n - E_m|T \geq \hbar \), the analogue of (17). However, this time Heisenberg draws a different conclusion, namely, that the phase \( \varphi = 2\pi(E_n - E_m)T/\hbar \), connected with the transition between the stationary states \( E_n \) and \( E_m \), is uncertain by an amount \( \Delta\varphi \geq 2\pi \), and he concludes that as soon as the two beams can be separated “the phases are completely indeterminate (‘unbestimmt’)”. Clearly, the suggestion here is that this is an example of the uncertainty principle between energy and phase. However, the phase \( \varphi \) is quite different from the phase variables \( \psi \) appearing in (16): \( \varphi \) is not a quantum mechanical observable. Moreover, it comprises both ‘uncertainties’ appearing in (17). What the Stern–Gerlach example does show is that it takes (external) time \( T \) to separate states with energy \( E_n \) and \( E_m \), but this is not an illustration of Eq. (16), which refer to a single instant of time.

In Section 3 of the article a surprise is waiting for us. After having given the quantum-mechanical equations of motion \( \dot{p} = -\partial H/\partial q \), \( \dot{q} = \partial H/\partial p \), and having applied them to a free particle: \( H = p^2/2m \), \( \dot{q} = p/m \), \( \dot{p} = 0 \), Heisenberg continues:

As time can be treated as a parameter (or a “c-number”) when there are no time-dependent external forces, the solution of this equation is \( q = p_0 t/m + q_0; \ p = p_0 \).


This is a particularly strange passage since time-dependent forces did not enter the discussion of (16) either, and there, time was certainly not a c-number. Heisenberg seems to echo Dirac here, who, in his first article on time, took the presence of time-dependent external forces as a decisive reason for converting the time parameter into an observable of the system (Dirac, 1926a).

In Section 4, near the end of the article, the Stern–Gerlach experiment is considered again. The total wave function is a superposition of the wave functions of atoms with internal energies \( E_1 \) and \( E_2 \), respectively. Now the wave function of the atom is written as a product of the wave function of its centre of gravity and of its internal state and it is supposed that the wave function of the centre of gravity differs from zero only in a small region of space. Separation of the beams now means that the overlap of the two centre of gravity wave functions vanishes. As a consequence the interference term between the two beams disappears from the total probability distribution and, concludes Heisenberg, “with it the possibility of measuring a phase relation”. Of course, we are here again dealing with the phase of the wave functions, a quantity quite distinct from the phase angles \( \psi \) in (16). Moreover, the vanishing of the interference terms does not depend on the internal states of the atoms and is not at all peculiar to eigenstates of the internal energy.
We conclude that Heisenberg’s article is confused about the meaning of equations (16). As is shown in Appendix A, these equations can be interpreted as pertaining to dynamical variables of special systems: certain multiply periodic systems and clocks. Heisenberg, on the other hand, suggests in his article that Eq. (16) is relevant to all situations where energy, time and phases, in any meaning of these words, play a role. Possibly because of this vagueness, the first equation in (16) came to be separated from the second one and started a life of its own as a relation between energy and time in general. Insofar as such an equation exists it should have a different sign, as we have seen. This sign difference is not unimportant, because, under the pressure of relativity, Heisenberg’s equation \( Et - tE = -i\hbar \) came to be seen as the ‘natural’ relativistic counterpart for ‘time’ of the equation \( pq - qp = -i\hbar \) for ‘space’ (cf. Section 1.4). Because elementary quantum mechanics primarily deals with systems having position observables with eigenvalues continuously ranging over the whole real axis, \( q \) came to be seen as position in general, independent of any particular physical system, and the lack of an analogous operator for time became the ‘deep’ problem of time in quantum mechanics.

3.3. Bohr

Although the influence of Niels Bohr (1885–1962) on the interpretation of quantum mechanics is enormous, his views are of surprisingly little relevance to our subject, the problem of time in the early development of quantum mechanics. This is a consequence of the fact that in his interpretation of quantum mechanics the formalism of quantum mechanics plays a very subordinate role. Central to his interpretation is the concept of complementarity that developed from his intensive struggle with the wave–particle duality. About the same time that Heisenberg discovered the uncertainty relations Bohr’s ideas on complementarity had taken shape and his reaction to Heisenberg’s work was quite enthusiastic: he saw the uncertainty relations as the formal or ‘symbolic’ expression of his complementarity viewpoint. (He was less enthusiastic, however, about Heisenberg’s interpretation of the uncertainty relations.) An essential element of Bohr’s view is that the results of observations on atomic objects must, in the end, be expressed in the language of classical physics. However, not all classical concepts are simultaneously applicable. In particular, the description of a process as evolving in space–time, and the description based on the application of the conservation laws of energy and momentum, are mutually exclusive; these descriptions are complementary.

Bohr’s own derivation of the uncertainty relations (Bohr, 1928) starts from the relations \( E = \hbar \nu \) and \( p = \hbar /\lambda \) which connect the notions of energy \( E \) and momentum \( p \) from the particle picture with those of frequency \( \nu \) and wavelength \( \lambda \) from the wave picture. Now a wave packet of limited extension in space and time can only be built up by the superposition of a number of elementary waves with a large range of wave numbers and frequencies. Denoting the spatial and temporal extensions of the wave packet by \( \Delta x \) and \( \Delta t \), and the extensions in the wave number \( \sigma = 1/\lambda \) and frequency by \( \Delta \sigma \) and \( \Delta \nu \), it follows from Fourier analysis that in the most favorable case
$\Delta x \Delta \sigma \approx \Delta t \Delta \nu \approx 1$, from which one obtains the relations $\Delta t \Delta E \approx \Delta x \Delta p \approx \hbar$. Here, the symbols $\Delta x$, etc. denote unspecified measures of the size of a wave packet.

In contrast to Heisenberg, who associated the uncertainty relations with commutation relations between operators, Bohr subsumed them under the heading of complementarity. Thus, the quantities $\Delta x$, $\Delta t$, called ‘latitudes’ by Bohr, must in the first instance be interpreted as extensions of wave packets in ordinary space and time. This is particularly so in the case of $\Delta t$, as is evidenced by his various discussions of the relation $\Delta t \Delta E \approx \hbar$ (Bohr, 1928, 1949). Presumably, then, the problem of defining an operator for time did not arise with Bohr.

3.4. Schrödinger

Erwin Schrödinger (1887–1961) shared with Dirac a great interest in relativistic wave mechanics. His study of the Dirac equation and its physical meaning led to his discovery of the so-called *Zitterbewegung* (quivering motion) of the Dirac electron. If the variables $x_1$, $x_2$, $x_3$ of the Dirac wave function are interpreted as eigenvalues of the components of the position observable of the electron, the motion of a free electron turns out to be extremely complicated. Schrödinger showed that these variables can be written as the sum of two parts $x_k = X_k + \xi_k$ such that $X_k$ executes the normal uniform rectilinear motion while $\xi_k$ oscillates rapidly with amplitude $\sim \hbar/mc$ and frequency $\sim mc^2/\hbar$ (Schrödinger, 1930). This raised for him the question whether it is the positions $x_k$ or the positions $X_k$ that appear, together with the time parameter $t$, in the Lorentz transformation and, more generally, whether the basic ideas of special relativity are consistent with quantum mechanics (Schrödinger, 1931a):

Es erscheint also fraglich, ob als Lorentztransformation nach wie vor die wohlbekannte Substitution zwischen den Eigenwerten von $x_1$, $x_2$, $x_3$ und dem Parameter $t$, oder zwischen den Eigenwerten von $X_1$, $X_2$, $X_3$ und dem Parameter $t$ anzusehen ist. Ferner möchte ich darauf hinweisen, dass dem mathematisch so klaren und einfachen Begriff der Lorentztransformation vom quantentheoretischen Standpunkt aus eine recht erhebliche Schwierigkeit anhaftet. Man pflegt die Koeffizienten einer Lorentztransformation als genau bekannt vorauszusetzen. Damit setzt man die Relativgeschwindigkeit der zwei Koordinatensysteme als genau bekannt voraus. Dadurch wird aber, nach der Heisenbergschen Unschärferelation, ihre relative Lage vollkommen unbestimmt. Rein mathematisch kann man dieser Schwierigkeit natürlich entgehen, indem man sich zwei hinreichend massige und doch absolut starre physische Systeme denkt. Vom physikalischen Standpunkt wird aber zu bedenken sein, dass es derlei in Wirklichkeit nicht gibt und nicht geben kann. (Schrödinger, 1931a, p. 72.)

[Thus it seems questionable whether as usual the Lorentz transformation should be regarded to be the well-known substitution between the eigenvalues $x_1$, $x_2$, $x_3$ and the parameter $t$, or between the eigenvalues $X_1$, $X_2$, $X_3$ and the parameter $t$. I further want to point out that the mathematically clear and simple notion of the Lorentz transformation is stuck with a very considerable difficulty from a]
quantum mechanical standpoint. One usually takes the coefficients of a Lorentz transformation to be accurately known. Doing that, one supposes that the relative velocity of the two coordinate systems is accurately known. However as a consequence, by Heisenberg’s uncertainty relation, their relative position becomes totally undetermined. Purely mathematically, of course, one can avoid this difficulty by imagining two sufficiently massive and yet absolutely rigid physical systems. From a physical point of view, however, one must realize that such systems do not and cannot exist. (Translation J.H.).]

Note how the distinction between the meaning of the $x_1$, $x_2$, $x_3$ as eigenvalues of the position of electrons and as the spatial coordinates of a frame of reference becomes blurred in this quotation (cf. Section 1.3).

The problem of the Lorentz transformation is taken up again in the article “Spezielle Relativitätstheorie und Quantenmechanik” (Schrödinger, 1931b). Schrödinger reviewed his work in relativistic electron theory in an article in the Annales de l’Institut Henri Poincaré (Schrödinger, 1932). The following summary is based on these two articles.

The first question Schrödinger turns to is the problem of setting up a space–time coordinate system as required by special relativity theory. He considers the synchronization of a clock by means of light signals. To achieve an accuracy $\tau$ in time, the frequency spread of the signal must at least be of order $1/(4\pi\tau)$; hence the recoil momentum transferred to the clock is uncertain by an amount $h/(4\pi\tau\varepsilon)$. This implies an uncertainty in the velocity of the clock, and, by the laws of special relativity, an uncertainty in the rate of the clock. This uncertainty adds on to the uncertainty $\tau$. Schrödinger shows that the resulting total uncertainty in the adjustment of a clock of mass $m$ cannot be smaller than $h/(4\pi\varepsilon m c^2)$. Similar considerations show that a measurement of the spatial distance between two points of an object of mass $m$ is uncertain by at least $h/(4\pi\varepsilon mc)$. This means that the setting up of an accurate space–time coordinate system requires the use of very massive clocks and rods. Schrödinger concludes that from the point of view of quantum mechanics special relativity, like classical mechanics, is to be considered as only a macroscopic approximation. It seems likely, then, that formulas from special relativity, such as the Lorentz transformation, cannot simply be copied in quantum mechanics; they should be ‘quantized’. According to Schrödinger, this must mean that the coordinates appearing in the classical Lorentz transformation should be replaced by operators. But now we meet with a serious problem: whereas classically the spatial coordinates and the time coordinate are on an equal footing—the basic idea of the Lorentz transformation—in quantum mechanics this is not so. Time in quantum mechanics is not an operator but a $c$-number; its value is supposed to be exactly known. In Schrödinger’s words it is “the good old time of Newton and quantum mechanics has no problem with the existence of the good old mantel clock that it needs to know the value of the parameter $t$.” Schrödinger considers this as a very serious defect of quantum mechanics and not only because of relativity: knowledge about $t$ is acquired in the same way as knowledge about any other physical variable by observing a certain physical system, namely, a clock. Therefore,
$t$ is an observable and must be treated as such. There is no justification for the exceptional role of the time. Schrödinger goes on to examine the properties of an ideal clock. To do this it is necessary to assume the existence of two such clocks, the first serving to define the time $t$ appearing in the wave function describing the configuration of the second clock, e.g. the position of its pointer. For the second clock to be ideal, wave functions at different values of $t$ must be orthogonal. Schrödinger shows that this can only be the case if the energy of the clock is totally uncertain: all values of the energy must be equally probable (cf. Appendix A). He concludes:

Ein solcher Systemzustand ist wohl physikalisch sinnlos; die Idealuhr, von welcher die Q.M. durch Verwendung der Variablen $t$ Gebrauch macht, ist mit den Grundlagen der Q.M. im Widerspruch. (Schrödinger, 1931b, p. 247.)

[Such a state of the system is physically meaningless; the ideal clock, that quantum mechanics uses by the application of the variable $t$, is in contradiction with the foundations of quantum mechanics. (Translation J.H.).]

Schrödinger was the first to explicitly consider time as defined by a clock in quantum mechanics. However, to investigate the properties of such a clock a first clock is needed to define what I have called the external time parameter. The performance of the second clock, defining an internal time, is judged by the accuracy with which it mimics the external time. Thus, Schrödinger comes close to solving the problem in the way I have suggested, but he is stopped by two problems. First, there is the difficulty of the existence of an ideal quantum clock, a problem that I believe is not fundamental (cf. Pauli and Appendix A). An ideal quantum clock is an idealization which can be arbitrarily approximated. In a sense, an ideal clock is the analogue for time of a point particle, an ideal indicator of position. Both idealizations imply infinitely extended spectra of conjugate variables, energy in the first case, momentum in the second. Schrödinger’s second problem concerns the external unquantized space–time background represented by his first clock. In the usual quantum mechanics the presence of an unquantized external space–time reference system is taken for granted (cf. Section 1.1) and the existence of a classical level of description even seems basic to Bohr’s interpretation of quantum mechanics. However, not everybody accepts Bohr’s interpretation and in the theory of quantum gravity the problem of the background space–time becomes acute. Thus Schrödinger’s second problem remains unsettled.

3.5. Von Neumann

John von Neumann (1903–1957) developed the mathematical foundations of quantum mechanics in his celebrated book *Mathematische Grundlagen der Quantenmechanik* (von Neumann, 1932, 1955). In Chapter V of the book the problem of measurement in quantum mechanics is discussed and von Neumann presents his (in)famous two manners of evolution of a physical system: first, if a

---

measurement is performed on the system its state changes in a discontinuous way; second, if no measurement is performed, the state evolves continuously according to the Schrödinger equation. Von Neumann stresses the fact that the first change must be instantaneous, a fact which he finds questionable because:

it is well-known that there is a quantity which, in classical mechanics, is canonically conjugate with time: the energy. Therefore it is to be expected that for the canonically conjugate pair time–energy, there must exist indeterminacy relations similar to those of the pair cartesian coordinate–momentum. Note that the special relativity theory shows that a far reaching analogy must exist: the three space coordinates and time form a “four vector” as do the three momentum coordinates and the energy. Such an indeterminacy relation would mean that it is not possible to carry out a very precise measurement of the energy in a very short time. In fact, one would expect for the error of measurement (in the energy) and the time duration $\tau$ a relation of the form $\varepsilon \tau \sim h$. (von Neumann, 1955, p. 353.)  

This interpretation of the uncertainty relation for energy and time is wrong, as we have seen, but, apparently, this interpretation was suggested to von Neumann by Heisenberg’s book (Heisenberg, 1930) to which, at this point, he refers.

After having discussed the relation between the energy $\varepsilon = hv$ and duration $\tau$ of a light quantum as an example of the relation $\varepsilon \tau \sim h$, von Neumann asks himself: “Then how can our assumption of instantaneous measurements be justified?”

We have seen that Schrödinger was of the opinion that the coordinates $x$, $y$, $z$, $t$ appearing in a Lorentz transformation should be operators in quantum mechanics, but that this requirement cannot be fulfilled since (or, so long as) the time $t$ is a $c$-number. Von Neumann expresses the same opinion:

First of all we must admit that this objection [against the possibility of instantaneous measurements] points to an essential weakness which is, in fact, the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes the time $t$ from the three space coordinates $x$, $y$, $z$, and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those $x$, $y$, $z$ closely connected with $t$ by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter $t$, just as in classical mechanics. Or: a system consisting of 2 particles has a wave function which depends on its $2 \times 3 = 6$ space coordinates, and only upon one time $t$, although, because of the Lorentz transformation, two times would be desirable. It may be connected with this non-relativistic character of quantum mechanics that we can ignore the natural law of minimum duration of the measurements. This might be a clarification, but not a happy one! (von Neumann, 1955, p. 354.)

Here, we again encounter the confusion between the coordinates of a point of space and the position coordinates of a particle, a confusion that is clearly fostered by the notation $x$, $y$, $z$ for both concepts. In quantum mechanics the coordinates of a point of space are $c$-numbers, just like the time parameter $t$, and the Lorentz transformation causes no problem.
3.6. Pauli

Wolfgang Pauli (1900–1958) is the author of three famous comprehensive review articles: on relativity theory (1921), on the old quantum theory (1926), and on the new quantum mechanics (1933). He begins his encyclopedia article on the new quantum mechanics in the spirit of Bohr with a discussion of complementarity and the indeterminacy relations by considering the wave–particle duality of light (Pauli, 1933, 1980). The uncertainties simply are the extensions $\Delta x$ and $\Delta t$ of wavepackets in space and time. Next, the accuracy is considered with which the location in space and time of a material particle can be determined by scattering light from it. The maximum accuracy of the position measurement is found to be $\Delta x \sim \lambda_m v/c$, where $v$ is the velocity and $\lambda_m$ the de Broglie wavelength $\hbar/|p|$ of the particle. The minimum time needed for the position measurement (i.e. for the scattering process) is $\Delta t \sim \Delta x/c$. Hence, in non-relativistic quantum mechanics, it is meaningful to neglect $\Delta t$ when $\Delta x$ is given. We thus arrive at the following basic assumption of non-relativistic quantum theory:

\[
W(x_1, x_2, x_3; t) \, dx_1 \, dx_2 \, dx_3
\]
dafür, dass das Teilchen sich innerhalb des Spielraums $(x_1, x_1 + dx_1; x_2, x_2 + dx_2; x_3, x_3 + dx_3)$ am Ort $x_1, x_2, x_3$ befindet. (Pauli, 1933, p. 92.)

[In every state of a system, first of all for a free particle, there exists, at each instant of time $t$, a probability $W(x_1, x_2, x_3; t) \, dx_1 \, dx_2 \, dx_3$ of finding the particle in the volume interval $(x_1, x_1 + dx_1; x_2, x_2 + dx_2; x_3, x_3 + dx_3)$ at the place $x_1, x_2, x_3$. (Translation J.H.).]

In jedem Zustand eines Systems, zunächst bei einem kraftfreien Teilchen, existiert in jedem Zeitmoment $t$ eine Wahrscheinlichkeit $W(x_1, x_2, x_3; t) \, dx_1 \, dx_2 \, dx_3$ dafür, dass das Teilchen sich innerhalb des Spielraums $(x_1, x_1 + dx_1; x_2, x_2 + dx_2; x_3, x_3 + dx_3)$ am Ort $x_1, x_2, x_3$ befindet. (Pauli, 1933, p. 92.)

An analogous assumption is shown to hold for the momenta of free particles. Pauli remarks that in the formulation of the basic assumption time is distinguished from position since the time coordinate is supposed to be exactly given. In a footnote he refers to the article by Schrödinger we have discussed:

Auf diesem Umstand ist besonders von E. Schrödinger (Berl. Ber. 1931, S. 238) hingewiesen worden. In diesem Zusammenhang wird dort auch betont, dass eine ideale, d.h. die Zeit exakt angebende Uhr, eine unendlich grosse Energieunsicherheit, also auch eine unendlich grosse Energie besitzen würde. Nach unserer Meinung bedeutet das allerdings nicht, dass die Benutzung des gewöhnlichen Zeitbegriffes in der Quantenmechanik widerspruchsvoll sei, da eine solche ideale Uhr beliebig angenähert werden kann. Man denke sich z.B. einen sehr kurzen (im Limes unendlich kurzen) Lichtwellenzug, der (infolge des Vorhandenseins geeigneter Spiegel) einen geschlossenen Weg beschreibt. (Dabei bleibt allerdings, wie im Text bereits hervorgehoben, die Frage der Existenz solcher Spiegel noch ausser Diskussion.) (Pauli, 1933, p. 92.)

[In particular E. Schrödinger (1931b, p. 238) has drawn attention to this fact. In this connection, it is also stressed there that an ideal clock, i.e. one which gives the time exactly, will possess an infinitely large uncertainty in energy and hence also an infinite energy. According to us, this does not mean that the use of the usual
concept of time is inconsistent with quantum mechanics, since such an ideal clock can be approximated arbitrarily. One can imagine, e.g., a very short (in the limit, infinitesimally short) wave train of light, which due to the presence of suitable mirrors describes a closed path. (But the question of the existence of such mirrors is outside our purview, as already mentioned in the text.) (Translation J.H.).]

(This view of Pauli’s on the existence of an ideal clock is corroborated by our example in Appendix A.)

From this point the formalism of non-relativistic quantum mechanics for particles with mass is developed. The wave function of a system of $f$ particles is denoted by $\psi(q_1, \ldots, q_f; t)$, with $t$ the time parameter. In Section 8 the equation of motion in the Heisenberg representation is derived:

$$i\hbar \frac{dF}{dt} = FH - HF,$$

where the operator $F$ is an arbitrary function of the $q$’s and $p$’s not containing the time explicitly. To this formula is added the following footnote on the existence of an operator of time:

In der älteren Literatur über Quantenmechanik findet sich an Stelle von [(18)] oft die Operatorgleichung

$$Ht - tH = (\hbar/\imath)1,$$

die aus [(18)] formal durch Einsetzen von $t$ für $F$ entsteht. Es ist indessen im allgemeinen nicht möglich, einen Hermiteschen Operator (z.B. als Funktion der $p$ und $q$) zu konstruieren, der diese Gleichung erfüllt. Dies ergibt sich schon daraus, dass aus der angeschriebenen V.R. gefolgert werden kann, dass $H$ kontinuierlich alle Eigenwerte von $-\infty$ bis $+\infty$ besitzt (vgl. Dirac, Quantenmechanik (1930), S. 34 u. 56), während doch andererseits diskrete Eigenwerte von $H$ vorkommen. Wir schliessen also, dass auf die Einführung eines Operators $t$ grundsätzlich verzichtet und die Zeit $t$ in der Wellenmechanik notwendig als gewöhnliche Zahl ("c-Zahl") betrachtet werden muss (vgl. hierzu auch E. Schrödinger, Berl. Ber. (1931) S. 238). (Pauli, 1933, p. 140.).

[In the older literature on quantum mechanics, we often find the operator equation $Ht - tH = (\hbar/\imath)1$, which arises from [(18)] formally by substituting $t$ for $F$. It is generally not possible, however, to construct a Hermitian operator (e.g. as a function of $p$ and $q$) which satisfies this equation. This is so because, from the commutation relation written above, it follows that $H$ possesses continuously all eigenvalues from $-\infty$ to $+\infty$ (cf. Dirac, 1930, p. 34 and 56), whereas on the other hand discrete eigenvalues of $H$ can be present. We, therefore, conclude that the introduction of an operator $t$ must fundamentally be abandoned and the time $t$ must necessarily be considered as an ordinary number ("c-number") in wave mechanics (cf. for this Schrödinger, 1931b, p. 238). (Translation J.H.).]

This is the famous footnote to which almost all later discussions of the problem of time refer. Let us carefully examine it. It is true that, in a given system, it is generally not possible to construct from the $p$’s and $q$’s a Hermitian operator satisfying relation (19). However, it is not true that from (19) it follows that $H$ possesses
continuously all eigenvalues from $-\infty$ to $+\infty$. In Appendix A we discuss a system with operators $t$ and $H$ satisfying (19), where $H$ has discrete eigenvalues ranging from $-\infty$ to $+\infty$. Apart from this we shall argue that it is not unphysical for an isolated system to possess continuous energy eigenvalues ranging from $-\infty$ to $+\infty$. But these are not the points we want to make here. What the first part of the footnote says is that relation (19) cannot be satisfied by each and every Hamiltonian operator $H$. This is true if $t$ is the internal time of some physical system. However, the last, italicized sentence suggests a switch from internal time operators to time in general, and in this general sense the sentence has been understood by many in later years.

We conclude that, whereas Pauli saw the existence of material clocks in quantum mechanics as unproblematic (first footnote), the idea of introducing a time operator $t$ must be given up (second footnote). This can only mean that Pauli interpreted $t$ as time in general, i.e. time not connected with some specific physical system. But, as we have stressed several times, neither time ‘in general’, nor space ‘in general’, is represented by an operator in quantum mechanics. Because of the abundance of position operators $q$ in physics, these operators became confused with position ‘in general’, and the problem of the ‘missing’ general time operator came into being.

3.7. Conclusion

Of the six authors considered in this article, Dirac most explicitly tried to incorporate the external time parameter as an operator in quantum mechanics. His goal was to extend Heisenberg’s matrix mechanics to a relativistic theory. He gave up this attempt, temporarily, when he developed his non-covariant general theory of operators and states, and returned to the problem of relativistic quantum mechanics only in the case of a single particle. However, his relativistic electron wave equation soon became reinterpreted as an equation for an operator field in spacetime, and the electron position coordinates, now interpreted as the coordinates of a point of space, were combined with the time parameter to form the $c$-number coordinates of a point of spacetime. In the several editions of his Principles of Quantum Mechanics mention is no longer made of a time operator. Remarkably, Dirac never seemed to have contemplated the notion of internal time in quantum mechanics.

Heisenberg, in his 1927 paper, was primarily interested in demonstrating his uncertainty relations in specific cases. His view on time remained quite unclear. In his example of energy determination by means of the Stern–Gerlach experiment, time is most naturally interpreted as external time. But in his example of the energy and phase of the state of an atom, the phase may be viewed as an internal time.

Bohr does not seem to have had any trouble with time in quantum mechanics, possibly because he was not very much interested in the problem of how to represent classical quantities in quantum mechanics.

Schrödinger took the point of view that time, in the last instance, is defined by a clock. For him time is what we have called ‘internal’ time. He found that an ideal clock, in quantum mechanics, must have an energy spectrum ranging from plus to minus infinity and he concluded that, from a physical point of view, such a system cannot exist. Since relativity theory needs ideal clocks to set up a frame of reference
in spacetime, a serious clash between relativity and quantum mechanics arises. In his discussion of the Lorentz transformation the distinction between internal and external space and time becomes blurred.

Von Neumann saw it as the chief weakness of quantum mechanics that the external time parameter is not quantized. The space coordinates appearing in the Lorentz transformation are interpreted by him as position variables of particles, i.e. as internal variables.

Pauli mentioned the problem of time in two footnotes in his encyclopedia article, in the first of which he refers to the work of Schrödinger. Pauli saw no fundamental problem connected with the usual concept of time in quantum mechanics since an ideal clock can be approximated arbitrarily. But in the second footnote he concluded that the introduction of an operator for \( t \) is not possible. Apparently, \( t \) must be interpreted, here, as an unspecified external time parameter. I have argued that the \( c \)-number character of the external time parameter is not problematic since the external space coordinates also are \( c \)-numbers in quantum mechanics. However, Pauli’s conclusion, italicized by himself, suggests that something quite remarkable is amiss, and this footnote has been interpreted in that way by many later authors.

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Appendix A

In this appendix some simple examples of angle and action variables are discussed, first classically and then in quantum mechanics, to elucidate the discussion in the text. A possible interpretation of Heisenberg’s equations (16) is given and the remarks of Schrödinger and Pauli about ideal quantum clocks are illustrated.

First, consider a system characterized by an angle variable \( w \) and an action variable \( J \), both ranging over the whole real axis. We have the canonical equation \( \{ w, J \} = 1 \) and the equations of motion \( \frac{dw}{dt} = \{ w, H \} \), \( \frac{dJ}{dt} = \{ J, H \} \), where \( H \) is the Hamiltonian, a function of \( J \) only. As a simple choice, consider \( H = vJ \) with \( v \) a constant. Then, \( \frac{dw}{dt} = v \), \( \frac{dJ}{dt} = 0 \), with the solutions \( w(t) = w_0 + vt \), \( J(t) = J_0 \), where \( w_0 \) and \( J_0 \) are arbitrary constants. The variable \( \tau = w/v \) satisfies \( \frac{d\tau}{dt} = \{ \tau, H \} = 1 \), that is, \( \tau(t) = \tau_0 + t \). Hence, \( \tau \) behaves exactly like the time parameter \( t \) and the system can be looked upon as an ideal linear clock. However, the conceptual difference between \( \tau \), a dynamical variable, and \( t \), the time parameter, should be noted.

Now suppose that the system is periodic, that is, all dynamical variables are periodic functions of \( w \) with period \( 2\pi \). We may also think of \( w \) as being confined to the interval \([0, 2\pi]\). The equations are unchanged but \( w \) now is a
dimensionless angle variable, also called phase angle or phase, \( \nu \) is an angular frequency, \( J \) has the dimension of an action (energy \( \times \) time) and \( \tau = \nu / \nu \) is a periodic time-variable.

Next, consider a collection of such systems described by canonical variables \( w_1, \ldots, w_n, J_1, \ldots, J_n \), frequencies \( \nu_1, \ldots, \nu_n \), and Hamiltonian \( H = \sum_i \nu_i J_i \). We have

\[
\{w_i, J_j\} = \delta_{ij}, \quad dw_i/dt = \{w_i, H\} = \nu_i, \quad dJ_i/dt = \{J_i, H\} = 0.
\]

There also are \( n \) time-variables \( \tau_i = w_i / \nu_i \) and, since the frequencies are independent of the \( J \)'s, the variables \( \tau_i = w_i / \nu_i, \eta_i = \nu_i J_i \) form a new set of canonical variables satisfying the simple equations

\[
\{\tau_i, \eta_j\} = \delta_{ij}, \quad d\tau_i/dt = \{\tau_i, H\} = 1, \quad d\eta_i/dt = \{\eta_i, H\} = 0 \quad \text{with} \quad H = \sum_i \eta_i.
\]

In more general cases the frequencies do depend on the \( J \)'s, and then there is no canonical transformation leading from the action and angle variables to pure time-variables.

Simple as they are, these examples illustrate important facts. First, they show the difference between the time parameter \( t \) on the one hand, and the phase and time variables on the other. Second, they show the difference between the Hamiltonian and the action variables. The action variables are the canonical conjugates of the phase and time variables, the Hamiltonian is the unique generator of the motion.

In quantum mechanics all dynamical variables are replaced by self-adjoint operators (printed bold face) and all Poisson Brackets are replaced by commutators according to the substitution \( \{,\} \rightarrow (i\hbar)^{-1}[.,.] \).

In our first example, this leads to the relation \( \{w, J\} = i\hbar \mathbf{1} \), analogous to the relation \( [q, p] = i\hbar \mathbf{1} \) for a free particle, and with the same solution: in the representation where \( w \) is diagonal the Hilbert space is formed by the square-integrable functions \( f(w) \) defined on the whole real \( w \)-axis and \( J \) is the operator \( J = -i\hbar d/dw \). However, the Hamiltonian \( H = \nu J \) differs from the Hamiltonian of a free particle: \( H_{\text{particle}} = p^2 / 2m \). In particular, the latter is positive definite, whereas the eigenvalue spectrum of \( H = \nu J \) is the whole real axis.

In our second example, the Hilbert space in the \( w \)-representation is formed by the square-integrable functions \( f(w) \) on the interval \( 0 \leq w \leq 2\pi \). The angle and action operators are again represented by multiplication and differentiation operators:

\[
w f(w) = w f(w), \quad J f(w) = -i\hbar d f(w)/d w,
\]

satisfying the commutation relation

\[
w J - J w = i\hbar \mathbf{1}.
\]

The operator \( w \) is self-adjoint on the whole Hilbert space, while \( J \) is self-adjoint on the subspace of the square-integrable, differentiable functions satisfying \( f(0) = f(2\pi) \). These operators have complete, orthonormal sets of generalized
eigenstates $|w\rangle$ and $|m\rangle$:

$$w |w\rangle = w |w\rangle, \quad \langle w | w' \rangle = \delta (w - w'),$$
$$J |m\rangle = m |m\rangle, \quad \langle m | m' \rangle = \delta_{m,m'},$$

where the eigenvalue $w$ runs through the interval $[0, 2\pi]$ and $m = 0, \pm 1, \pm 2, \ldots$.

In the $w$-representation the states $|w\rangle$ and $|m\rangle$ are represented by the wave functions

$$|w\rangle = \frac{1}{\sqrt{C_0}} e^{i m w} |m\rangle, \quad |m\rangle = \frac{1}{\sqrt{C_0}} e^{i w m} |w\rangle,$$

with $C_0 = \sqrt{\int_0^{2\pi} e^{-i m w} |m\rangle |w\rangle dw}$.

This implies the uncertainty principle for $J$ and $w$. In an eigenstate of $J$ the phase angle $w$ is completely undetermined, and in an eigenstate of the phase the action is completely undetermined.

**Remark 1.** An operator is fully defined only if its domain (the subset of the Hilbert space on which it is defined) is specified. Thus, an operator equation like (A.1) needs the specification of the domain on which both sides are defined. The above solution of Eq. (A.1), where the eigenvalues of $w$ are confined to a finite interval, seems to contradict the Stone–von Neumann theorem, which is often thought to imply that two self-adjoint operators $a$ and $b$ satisfying the commutation relation $[a, b] = i \hbar \mathbf{1}$ must both be unbounded. However, the Stone–von Neumann theorem has the proviso that the commutator can be expressed in the so-called Weyl form. In the above representation this is not possible.\(^{10}\)

The operator of translations in time is $U(t) = e^{-i H t}$. With (A.2) we find

$$U(t) |w\rangle = e^{-i H t} |0, 1\rangle |w\rangle = (2\pi)^{-1/2} \sum_{m=-\infty}^{+\infty} e^{-i m w - i m n t} |m\rangle = |w + vt\rangle,$$

in agreement with what one would expect. Alternatively,

$$w(t) = e^{i H t} w_0 e^{-i H t} = w_0 + v t \mathbf{1}. \tag{A.3}$$

This result also follows from $dw/dt = -i [w, H] = -iv |w, J\rangle = v \mathbf{1}$. It is interesting to note that the phase angle has two parts: a time-independent part $w_0$, which is subject to the uncertainty principle and is completely uncertain in an eigenstate of $J$, and a time-dependent part that is a multiple of the identity operator with no uncertainty at all.

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\(^{10}\)For more details see Putnam (1967); Richtmyer (1978, 14.6); Uffink (1990, 2.6.2).
Again, a time operator \( \tau = \frac{w}{v} \) can be defined. The commutator of \( \tau \) and the Hamiltonian is simply

\[
\tau H - H \tau = i\hbar 1,
\]

where \( \hbar \) has been reintroduced. For a multiply periodic system we have, multiplying with a minus sign for easy comparison with Heisenberg’s (16):

\[
H \tau_i - \tau_i H = -i\hbar 1, \quad J_i w_j - w_j J_i = -i\hbar \delta_{ij} 1.
\]

(A.4)

Written this way, the meaning that may be given to Heisenberg’s equations (16), becomes clear.

**Remark 2.** In elementary discussions, an uncertainty relation between energy and time is sometimes derived from simple Fourier analysis of a wave train (cf. Bohr). If the wave train has length \( \Delta t \) in time its frequency spread \( \Delta \nu \) cannot be smaller than is allowed by the relation \( \Delta t \Delta \nu \approx 1 \). Using the relation \( E = h \nu \), one obtains \( \Delta \nu \Delta E \approx h \). These results are often interpreted as saying that it makes no sense to speak of the frequency of a vibration at a single instant of time, since ‘obviously’ an accurate determination of frequency needs a long time interval, and that the same, therefore, must hold for the determination of the energy of a system. Thus Heisenberg, in a passage already quoted in the text:

> The Stern–Gerlach experiment allows one to determine the magnetic or an average electric moment of the atom, and therefore to measure quantities which depend only on the action variable \( J \). The phases remain undetermined in principle. It makes as little sense to speak of the frequency of a light wave at a definite instant as of the energy of an atom at a definite moment. Correspondingly, in the Stern–Gerlach experiment the accuracy of the energy measurement decreases as we shorten the time during which the atom is under the influence of the deflecting field. (Heisenberg, 1927, p. 178; Wheeler and Zurek, 1983, p. 67.)

The first part of this quotation is true: our examples show that \( w \) is completely undetermined when the value of \( J \) is determined. However, it is an assumption of quantum mechanics that all observables, including \( J \) and the energy, can be precisely measured in an arbitrarily short time. This has nothing to do with the phases being undetermined if \( J \) is known. What about the frequency? Indeed, how long does it take to determine the frequency of a vibration? That depends on what you know about it. An arbitrary vibration may have no frequency at all so you cannot measure ‘its frequency’ in any amount of time! On the other hand, if you know that the vibration is sinusoidal its frequency can be determined in an arbitrarily short time. In our example the system is supposed to be (multiply) periodic with given frequency(ies). But the fixed frequency does not prevent the energy of the system from showing quantum uncertainty.

**Remark 3.** The above examples corroborate Schrödinger’s result that the energy spectrum of an ideal clock must range from \(-\infty\) to \(+\infty\), although the spectrum need not be continuous. It is often supposed that such a spectrum is not allowed on the physical ground that the system will act as an infinite source of energy. However,
as long as the system is isolated its energy is conserved and nothing of the sort will happen. Its normalizable states have finite (mean) energy that is a constant of the motion. In any case, our examples provide appropriate illustrations of the working of the formalism of quantum mechanics. Moreover, following Pauli’s suggestion in his first mentioned footnote, an ideal clock can be arbitrarily approximated by confining its energy spectrum to a large but finite interval (cf. Hilgevoord, 2002). Similar remarks can be made with regard to phase variables. An ideal angle operator \( w \) does not exist in a system if the corresponding action operator is non-negative, \( J \geq 0 \), as often is the case. This is probably the reason why action and angle variables do not play the same prominent role in quantum mechanics as they do in the older atomic theory. Finally, we note that if in (A.2) \( m \) is confined to values \( m \geq 0 \), the system becomes equivalent to a harmonic oscillator. However, the resulting mutilated states “\( \langle w \rangle \)” are no longer orthogonal and there is no ideal phase or time operator in this case.11

References


11 For detailed discussions of quantum clocks see the articles by Busch and by Mayato, Alonso and Eguquisquiza in Muga et al. (2002).


