

## **Der Geist bewegt die Materie... oder was ?**

Let me quote from:

G. 't Hooft, *Introduction to General Relativity*, 15 November 2010, pp. 25-26 (emphasis added) [http://www.staff.science.uu.nl/~hooft101/lectures/genrel\\_2010.pdf](http://www.staff.science.uu.nl/~hooft101/lectures/genrel_2010.pdf)

An important theorem is that the Riemann tensor *completely* specifies the extent to which space or space-time is curved, if this space-time is simply connected. We shall not give a mathematically rigorous proof of this, but an acceptable argument can be found as follows. Assume that  $R^\nu_{\kappa\lambda\alpha} = 0$  everywhere. Consider then a point  $x$  and a coordinate frame such that  $\Gamma^\nu_{\kappa\lambda}(x) = 0$ . We assume our manifold to be  $C_\infty$  at the point  $x$ . Then consider a Taylor expansion of  $\Gamma$  around  $x$ :

$$\Gamma^\nu_{\kappa\lambda}(x') = \Gamma^{[1]\nu}_{\kappa\lambda,\alpha}(x')^\alpha + \frac{1}{2}\Gamma^{[2]\nu}_{\kappa\lambda,\alpha\beta}(x')^\alpha(x')^\beta + \dots, \quad (5.33)$$

From the fact that (5.27) vanishes we deduce that  $\Gamma^{[1]\nu}_{\kappa\lambda,\alpha}$  is symmetric:

$$\Gamma^{[1]\nu}_{\kappa\lambda,\alpha} = \Gamma^{[1]\nu}_{\kappa\alpha,\lambda}, \quad (5.34)$$

and furthermore, from the symmetry (5.6) we have

$$\Gamma^{[1]\nu}_{\kappa\lambda,\alpha} = \Gamma^{[1]\nu}_{\lambda\kappa,\alpha}, \quad (5.35)$$

so that there is complete symmetry in the lower indices. From this we derive that

$$\Gamma^\nu_{\kappa\lambda} = \partial_\lambda \partial_\kappa Y^\nu + \mathcal{O}(x' - x)^2, \quad (5.36)$$

with

$$Y^\nu = \frac{1}{6}\Gamma^{[1]\nu}_{\kappa\lambda,\alpha}(x' - x)^\alpha(x' - x)^\lambda(x' - x)^\kappa. \quad (5.37)$$

If now we turn to the coordinates  $u^\mu = x^\mu + Y^\mu$  then, according to the transformation rule (5.5),  $\Gamma$  vanishes in these coordinates up to terms of order  $(x' - x)^2$ . So, here, the coefficients  $\Gamma^{[1]}$  vanish.

The argument can now be repeated to prove that, in (5.33), all coefficients  $\Gamma^{[1]}$  can be made to vanish by choosing suitable coordinates. Unless our space-time were extremely singular at the point  $x$ , one finds a domain this way around  $x$  where, given suitable coordinates,  $\Gamma$  vanish completely. All domains treated this way can be glued together, and only if there is an obstruction because our space-time isn't simply-connected, this leads to coordinates where the  $\Gamma$  vanish everywhere.

Thus we see that if the Riemann curvature vanishes a coordinate frame can be constructed in terms of which all geodesics are straight lines and all covariant derivatives are ordinary derivatives. This is a flat space.

G. 't Hooft, private communication, 26 Dec 2009: "I said, and I'll try repeat it transparently even to babies: here is the theorem: given ANY simply connected space-time or a simply connected subspace of space-time, and suppose that in all points  $x$  of that (sub) space the Riemann tensor vanishes entirely (zero, **zilch**, to use your mathematical vocabulary), then a coordinate frame exists such that in that space (I mean, in the very same space or subspace, clear?), the metric is flat, when expressed in that vary same coordinate frame. This means

that, in terms of those coordinates, we have  $g_{\mu\nu}(x) = \eta_{\mu\nu}(x)$ , in the usual terminology. Anything not clear yet about this statement?"

G. 't Hooft, private communication, 25 Dec 2009: "R\_{abcd}(x) = 0 FOR ALL VALUES OF ITS 4 INDICES, AT ALL POINTS x in **this** region of space-time. Is this clear enough?"

### Comments and questions

How was "**this** region of space-time" made separable from the rest of "regions" from the same spacetime ?

Mathematically, "if the Riemann tensor vanishes in the neighborhood of a point, then there will exist a chart at that point in which the Christoffel symbols vanish" (Adam Helfer, private communication). "As far as all the derivatives vanishing, if the curvature tensor vanishes in an open set, then all of its derivatives certainly vanish which in geodesic coordinates means the algebra will give the flat coordinates" (Maurice J. Dupre, private communication).

Physically, however, the situation is unclear. What will happen to the [Ricci tensor](#) at the *instant* at which the Riemann curvature tensor has totally vanished? What could **resurrect** the Riemann tensor from the dead flat space in those "glued" domains in which "**all** coefficients [XX] can be made to vanish by choosing suitable coordinates"? [Angels](#) ?

"Since the microwave background radiation (as well as [the "dark" energy](#) - D.C.) is everywhere and always, there can be nowhere where the Ricci tensor vanishes and therefore nowhere where the Riemann curvature tensor vanishes in a realistic physical model" (Maurice J. Dupre, private communication).

**Q1:** How would you resolve this paradox ?

The field equations of GR **imply** (see below) certain mathematical objects which can be rigorously defined only **locally**, yet they don't make sense *physically* (e.g., MTW, [p. 467](#)).

Surely one "may erect a locally inertial coordinate system in which matter satisfies the laws of special relativity" (Steven Weinberg, *Gravitation and Cosmology*, 1972, [pp. 62-68](#)), and claim that "particles follow "straight and uniform" inertial paths in each infinitesimal region of spacetime, and this in turn is a direct consequence of the **local** conservation of energy-momentum. It's true that the field equations of general relativity **imply** this conservation, as can be seen by the vanishing of the covariant divergence of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

"The field equations simply equate this to the energy-momentum tensor  $T_{mn}$ , so the covariant divergence of the latter must also vanish, hence energy-momentum is **locally** conserved, hence particles follow [geodesics](#)" (Kevin Brown, [General Relativity and the Principle of Inertia](#)).

The field equations of GR **imply** this conservation, but 'not in a local sense'. Excerpt from Harvey S. Reall (*General Relativity 2011*, [pp. 55-56](#)):

In GR (and SR) we assume that continuous matter always is described by a conserved energy-momentum tensor:

**Postulate.** The energy, momentum, and stresses, of matter are described by an *energy-momentum tensor*, a  $(0, 2)$  symmetric tensor  $T_{ab}$  that is *conserved*:  $\nabla^a T_{ab} = 0$ .

One might think that one could obtain global conservation laws in curved spacetime by introducing a definition of energy density etc for the gravitational field. This is a subtle issue. The gravitational field is described by the metric  $g_{ab}$ . In Newtonian theory, the energy density of the gravitational field is  $-(1/8\pi)(\nabla\Phi)^2$  so one might expect (e.g. from eq. (3.28)) that in GR the energy density of the gravitational field should be some expression quadratic in first derivatives of  $g_{ab}$ . But we have seen that we can choose normal coordinates to make the first partial derivatives of  $g_{ab}$  vanish at any given point. Gravitational energy certainly exists but not in a local sense.

**Q2:** Can we design a [quasi-local geodesic](#) with (i) "non-local" gravitational energy, (ii) "locally inertial coordinate system in which matter satisfies the laws of special relativity" (Steven Weinberg), and (iii) flux of energy and momentum from the gravitational field to the "nongravitational" (if any) matter (Hans Ohanian, [arXiv:1010.5557v1 \[gr-qc\]](#), p. 3)?

Hans C. Ohanian, arXiv:1010.5557v1 [gr-qc], p. 3

The universality of free fall motivates the geometric interpretation of gravity, with small test masses moving along geodesics of a curved spacetime geometry. In General Relativity, this geodesic motion can be shown to be a consequence of the “conservation” law  $T_{\mu}{}^{\nu}{}_{; \nu} = 0$  for nongravitational matter. But this is really a *nonconservation* law—it reveals to what extent the energy-momentum of the nongravitational matter is *not* conserved. Written out in full, it becomes

$$\frac{\partial}{\partial x^{\mu}} T_{\nu}{}^{\mu} = \Gamma_{\nu\mu}^{\alpha} T_{\alpha}{}^{\mu} - \Gamma_{\alpha\mu}^{\nu} T_{\nu}{}^{\alpha} \quad (1)$$

which determines the rate at which the nongravitational matter receives energy and momentum from the gravitational field (the equation is analogous to the equation for the rate of change of the momentum of a particle,  $dp/dt = F$ ).<sup>2</sup> Most writers on General Relativity fail to acknowledge that Eq. (1) is not so much a conservation law, as a law for energy transfer; Weinberg [(1972), p. 166] and Padmanabhan [2010, p. 213] are commendable exceptions. Only in a local geodesic coordinate frame (that is, in freely falling coordinates with  $\Gamma_{\mu\nu}^{\alpha} = 0$ ) does Eq. (1) reduce to the standard form of a conservation law,  $\partial T_{\nu}{}^{\mu} / \partial x^{\mu} = 0$ , which shows that the gravitational field delivers no energy or momentum to the nongravitational matter. From this we immediately recognize that the energy and momentum of a small test mass (for which energy contributions of order  $M^2$  can be neglected) are constant, so the body remains at rest in the freely falling coordinates and therefore moves along a geodesic.

I think we could have *quasi-local and non-linear*, due to gravity, interactions *only* in the case explained here. The crucial object, otherwise known as 'particles of the reference fluid', is Einstein's "total field of as yet unknown structure". We just can't paint a picture without a colorless canvas which defines all "colored" stuff "everywhere and for all time" (Ciufolini and Wheeler, p. 270) *dynamically* -- one-at-a-time, along the Arrow of Space.

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Thank you for your insights and professional comments.

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15 January 2012, 22:48 GMT

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